**TutorTube: Intro to Integration: Examples** Spring 2021

## Introduction

Hello! Welcome to TutorTube, where The Learning Center's Lead Tutors help you understand challenging course concepts with easy-to-understand videos. My name is Ebby, Lead Tutor for Math and Political Science. In today's video, we will have cover part two of the introduction to Integration with some examples. Let's get started!

## U-Substitution

U-sub is one of the more common methods that we utilize to find **the antiderivative or closed form solution** to some of the integrals that we will encounter. Let's practice it with the indefinite integral problem

$$\int \sin x \sin(\cos x) dx$$

The trick to finding the solution using u-sub is to determine the "best substitution" that would capture as much of the integral as possible. The right substitution might initially seem elusive to obtain but often times we only need to keep a few things in mind to solve it. First, let's let

$$u = \cos(x) \, .$$

The derivative of this is represented by du:

$$du = -sin(x)dx$$

Since we want our substitution to be as close to the original as possible let's factor out the negative

$$-du = \sin(x)\, dx$$

So, we can substitute the original equation with our new equation:

$$\int \sin(\cos x)\sin(x)dx = -\int \sin(u)\, du$$

Getting here makes it clearer to see why we chose u to be cos(x) in the first place, **it simplified the integral in such a way that made it immensely easier to find the antiderivative.** This is the essence of u-sub and we will explore more examples to get the hang of it. In the meantime, we turn our attention to solving this problem by asking ourselves "what derivative gives us **positive** sine?" The answer to that is **negative** cosine which means the solution is

$$-\cos(u)$$

and since u is cos(x)we substitute it back and we have

$$\cos(\cos x)$$

Do not forget to include what's called the constant of integration used for all indefinite integrals:

$$\boldsymbol{\cos(\cos x) + C}$$

Now let's apply U-sub to this problem

$$\int 2e^{cost} sintdt$$

A suitable U-sub would be u = cos(t) with du = -sin(t)dt. Thus, we'd re-write the integral as

$$-\int 2e^{u} du$$

As you may have noticed, it is necessary to format the integral in standard form in order to perform the integral correctly.

Now we use the fact that:

$$\int e^{x} dx = \frac{e^{x}}{x'}$$

To solve the integral:

$$-\int 2e^{u} du = -\left[\frac{2e^{u}}{u'}\right] = -2e^{u} + C$$

Finally, put the solution in terms of t

$$= \boldsymbol{-2e^{cost} + C}.$$

## Fundamental Theorem of Calculus (FTC)

There are two major parts in FTC. Part I basically highlights the inverse relationship between derivatives and integrals. In particular, if we take the derivative of a definite integral with lower bound of any constant and upper bound of an arbitrary x value, the solution is just the **integrand evaluated at the**

**upper limit** (of course remembering to apply chain rule if need be). We don't really focus too much on utilizing Part I in this course, but it is useful in other math courses like Probability where we may want to determine the derivative of continuous CDF function. Let's briefly practice Part I with the following functions:

1. $f(x) = \frac{d}{dx}\int_0^{x^5} cos\theta d\theta$
2. $f(x) = \frac{d}{dx}\int_{e^x}^{1} \ln t\, dt$

In the first example we see that the variable $x^5$ is in the correct position as the upper bound so we can obtain the derivative by "plugging in" $x^5$ into $\cos(\theta)$ and applying Chain Rule. First plug in:

$$\cos\,(x^5)$$

Then by Chain Rule, we multiply by the derivative of inner product $x^5$:

$$\mathbf{5x^4\cos\,(\boldsymbol{x^5})\,.}$$

In the second example we need to make sure that the variable is in the upper bound so we will utilize the fact that given any interval we can re-arrange the bounds as long as we put a negative out in front.

So, we have:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_{e^x}^{1} \ln t\, dt = -\int_1^{e^x} \ln t\, dt$$

Now apply FTC (remember chain rule)

$$-\ln e^x (e^x)'$$

Which simplifies to

$$\mathbf{xe^x}$$

## FTC Part II

Thus, we turn our attention to Part II of FTC which we utilize more frequently in single variable calculus. Let's find the integral of the following expressions

1. $f(x) = \int_1^{\sqrt{x}} \frac{z}{z^2+1} dz$
2. $\int_0^4 \frac{x}{\sqrt{1+2x}} dx$
3. $\int_0^1 (3t-1)^{60} dt$

FTC Part II States the following:

**The Fundamental Theorem of Calculus, Part 2** If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$, that is, a function such that $F' = f$.

Stewart, James *Calculus: Early Transcendentals*, 7th Ed., Brooks/Cole Cengage Learning, 2010

This means that **we must find the antiderivative first then plug b and a into it and subtract the two.** In the first example we will begin finding the antiderivative by utilizing the substitution techniques that we discussed previously. Let's take u to be

$$z^2 + 1$$

Which means that

$$du = 2zdz \to \frac{du}{2} = zdz$$

This illustrates that a good rule of thumb is to **take any constants along with your u-sub because they fall off after taking the derivative.** Thus, we can completely simplify the integral in terms of u as

$$\frac{1}{2}\int_1^{\sqrt{x}} \frac{du}{u},$$

One procedure for evaluating a definite integral using u sub is to change the bounds to be in terms of u. All this means is that we will use the u-sub that we created

$$u = z^2 + 1$$

and **plug in the lower and upper bound values into z.** Let's start with the lower bound of 1:

$$u(1) = 1^2 + 1 = 2$$

Therefore 2 is our new lower bound for the integral. The upper bound follows similarly

$$u(\sqrt{x}) = x + 1\,.$$

Now we are evaluating

$$\frac{1}{2}\int_{2}^{x+1} \frac{du}{u}\,du.$$

The benefit of doing this is that we **will not have to change the integral back in terms of x in order to evaluate the antiderivative.** To find the antiderivative of this integral **we need to know an important identity** that comes up often in calculus:

$$Since\,\frac{d}{du}\ln(u) = \frac{1}{u}\,, \int \frac{du}{u} = \ln\ |u|$$

Which means that we can **evaluate the antiderivative/integral:**

$$\left[\frac{1}{2}\ln|u|\right]_{2}^{x+1}.$$

Writing it in this format implies that we will use the FTC Part II formula to determine the solution. **We must always plug in the upper bound into all variables then plug in the lower bound and subtract the two.** First Evaluate the upper:

$$\frac{1}{2}\ln|x+1|.$$

Then evaluate the lower bound:

$$\frac{1}{2}\ln|2^2+1|$$

Simplify them and subtract

$$\frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|2|$$

Which simplifies by log rules to

$$\boldsymbol{\frac{1}{2}\ln\frac{x+1}{2} \rightarrow \ln\sqrt{\frac{x+1}{2}}}$$

In the second example, let's let

$$u = 1 + 2x$$

$$du = 2dx \ \rightarrow \frac{du}{2} = \mathrm{dx}$$

**In order to obtain the new bounds, plug in the given bounds into the x value of the substitution**. In this case, the lower bound is 0 so we'll plug it into x of the equation u = 1+2x.

$$u(0) = 1 + 2(0) \rightarrow 1$$

Doing the same thing for the upper bound of 4 yields

$$u(4) = 9\,.$$

Now we have

$$\frac{1}{2}\int_{1}^{9} \frac{x}{\sqrt{u}} du$$

o get the integral to be in terms of just u we will again use the original substitution equation

$$u = 1 + 2x\,.$$

If we re-arrange the equation, we can solve for x

$$u - 1 = 2x$$

$$\frac{u-1}{2} = x$$

**We now have x in terms of u and can complete the integral**

$$\frac{1}{2}\int_{1}^{9} \frac{u-1}{2\sqrt{u}} du$$

We should factor out ½ and split the integral

$$\frac{1}{2}\int_{1}^{9} \frac{1}{2}\left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du$$

Now we have

$$\frac{1}{4}\int_{1}^{9} u^{1-1/2} - u^{-1/2} du$$

Using the rules of exponents. We can now determine the antiderivative of the integral using **the reverse power rule (add one to the exponent and divide by the sum)**

$$= \frac{1}{4}\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^9$$

Be careful when simplifying and remember the rules of algebra that allow us to further simplify the function:

$$\frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^9$$

Now we are ready to apply FTC part two. First evaluate the upper bound

$$\frac{2}{3}(9)^{\frac{3}{2}} - 2(9)^{\frac{1}{2}} = 18 - 6$$

Then evaluate the lower bound

$$\frac{2}{3}(1^{\frac{3}{2}}) - 2(1^{\frac{1}{2}}) = 2/3 - 2$$

Subtract the two according the theorem:

$$18 - 6 \; - \left(\frac{2}{3} - 2\right) = \frac{40}{3}$$

And finally multiply it by the ¼ that we had on the outside

$$\frac{1}{4}\left[\frac{40}{3}\right] = \frac{10}{3}$$

Lastly for the third example, we will again solve the integral using u-substitution, however we will utilize the other method of computing defined integrals where we don't change the bounds. To begin let's let

$$u = 3t - 2$$

$$du = 3dt\,.$$

To get a perfect substitution let's divide du by 3.

$$du = 3dt \rightarrow \frac{du}{3} = dt$$

Now we can re-write the integral as

$$\frac{1}{3}\int_{0}^{1} u^{60}du\,.$$

We can now find the antiderivative of the integral by again applying **reverse power rule**

$$\frac{1}{3}\left[\frac{1}{61}u^{61}\right]_0^1$$

We can factor out 1/61

$$\frac{1}{183}[u^{61}]_0^1$$

Then change put u back in terms of x:

$$\frac{1}{183}[(3t-2)^{61}]_0^1$$

**Now,** we can evaluate the integral using FTC Part II

$$\frac{1}{183}(1-(-2)^{61})$$

Which simplifies to

$$\frac{1}{183}(1+2^{61})$$

As you may have surmised, it can be easy to mix up the two methods of solving definite integrals so you must be extra careful. **Either change the bounds right after you perform the U-sub or change the variables right before you apply FTC Part II, never both.**

## Additional Problems

Let's take a look at some additional problems

1. Find all functions $w$ if $w'(z) = \frac{15-z^2}{\sqrt{z}}$
2. 2. $\int \sec 6t\,(9\sec 6t - 7\tan 6t)\,dt$

In the first example we should first re-write the function

$$\frac{15}{\sqrt{z}}-\frac{z^2}{\sqrt{z}}$$

Then simplify

$$15z^{-\frac{1}{2}} - z^{2-\frac{1}{2}} = 15z^{-\frac{1}{2}} - z^{\frac{1}{2}}$$

Now take the integral because:

$$w(z) = \int w'(z) \rightarrow \int 15z^{-\frac{1}{2}} - z^{\frac{1}{2}}.$$

Then Apply the **Reverse power rule by adding one to the exponent and dividing by sum**

$$\int 15z^{-\frac{1}{2}} - z^{\frac{1}{2}} = \frac{15z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

Simplify

$$\frac{15z^{1/2}}{1/2} - \frac{z^{\frac{3}{2}}}{3/2} + C$$

And finally,

$$\boldsymbol{w(z) = 30z^{\frac{1}{2}} - \frac{2}{3}z^{\frac{3}{2}} + \mathrm{C}}.$$

In the second example we should first distribute the sec(6t):

$$\int 9\sec^2 6\,t - 7\sec 6t \tan 6t\, dt.$$

Then we can split up the integral **since it's separated by a subtraction (or addition)**

$$\int 9\sec^2 6\,tdt - \int 7\sec 6t \tan 6t\, dt$$

Now, we have two integrals for which we can evaluate independently. For the first integral we should realize that:

$$\frac{d}{dx}\tan(6t) = 6\sec^2(6t), so \int \sec^2 6\,tdt = \frac{1}{6}\tan(6t)\, dt$$

This is same principle that we used for the ln integral previously and it highlights the inverse relationship between derivatives and antiderivatives. Therefore,

$$\int 9\sec^2 6\,tdt = \frac{9}{6}\tan(x) + C = \frac{3}{2}\tan(x) + C.$$

The same principle can be used to solve the second integral because:

$$\frac{d}{dx}\sec(6t) = 6\sec(6t)\tan(6t)\ which\ means \int \sec 6t \tan 6t\, dt = \frac{1}{6}\sec(6t).$$

Thus,

$$\int 7\sec 6t \tan 6t\, dt = \frac{7}{6}\sec(6t) + C$$

Finally, we put the two solutions together according to the original integral and obtain

$$\boldsymbol{\frac{3}{2}tan(x) - \frac{1}{6}sec(6t) + C}\ (Remember\ any\ operation\ with\ C\ remains\ \text{"+C"})$$

## Outro

Thank you for watching TutorTube! I hope you enjoyed this video. Please subscribe to our channel @UNTLC for more exciting videos! Check out the links in the description below for more information about The Learning Center and follow us on social media. See you next time!