TutorTube: Number Systems

Fall 2020

Introduction

Hello and welcome to TutorTube, where The Learning Center's Lead Tutors help you understand challenging course concepts with easy to understand videos. My name is Jeff, Lead Tutor for Electrical Engineering. In today's video, we will explore Number Systems. Let's get started!

What Is A Number System?

What is a number system? A number system defines how many different numbers we can represent in a single digit. This is the same as saying how much stuff we can represent with a single number. Most places use **Base 10**, also called the decimal system. This means we can use the numbers **0** through **9** in a single digit's place. **Binary**, also known as **Base 2**, uses only **0** and **1** for a single digit's place. Likewise, **Base 8** uses **0** through **7**.

Base 16 means we can place **0** through **16** in a single digit's place. Because we do not normally have a single symbol to represent **10** through **15**, we use **A** through **F**. **A** represents **10**, **B** is **11**, and so on all the way to **F** is **15**.

Think of number systems as different languages that all use the same alphabet. The same symbols are present, but they have different meanings.

Polynomial Representation and Conversion to Decimal

Any number, regardless of the number system it is using, can be expressed as a polynomial weighting each power of the base against the number in each digit's place. For example, the number **154** can be broken up into

$$100 + 50 + 4$$

But if we chose to factor a power of **10** out of each term we get

$$1*10^2 + 5*10^1 + 4*10^0$$

Notice how the powers of **10** decrease by **1** as we move from the leftmost digit to the right. If we had a fractional part of the number, how would we represent that in a polynomial?

Let's say we have **2067.3** and we need to expand it into a polynomial form. Well we can apply the same pattern as before.

But what about the **.3**? If we follow the same pattern to extend past the decimal point, we'll write this as

$$2*10^3 + 0*10^2 + 6*10^1 + 7*10^0 + 3*10^{-1}$$

This same pattern applies to numbers of any base, not just **Base 10**. If we were given **221** in **Base 8**, we could expand it as

$$221_8 = 2*8^2 + 2*8^1 + 1*8^0$$

Notice how instead of each term including a power of **10**, now each term includes a power of **8** because the original number is in **Base 8**. Think about how this pattern relates to why we refer to the digit places as one's place, ten's place, etc.

We can use this polynomial expansion to allow us to convert numbers in any base to **Base 10**. Just by evaluating the polynomial, we'll get the equivalent number in **Base 10**. Take **1011** in **Binary** for instance. By evaluating this expression, it will collapse into **11**. This tells us that **1011** in **Binary** is **11** in **Decimal**. This also works for fractional numbers as long as we remember that negative exponents correspond to reciprocal numbers. Expanding **34.4** in **Octal**, which is **Base 8**, gives us 28.5 in **Decimal**. This applies to **Hexadecimal** as well. Remember that the capital letters **A**, **B**, etc. correspond to **10**, **11**, and so on. Because we're no longer restricted to a single digit place here, we can substitute **16** back into the expression. After simplifying we find that **BF6** in **Hexadecimal** is the same as **3062** in **Decimal**. There's no fancy tricks here, this will work for any base converting to **Base 10**.

Conversion to Any Base from Decimal

Converting to a base from **Decimal** is slightly more complicated. To convert to a new base, we repeatedly divide by the desired base and collect the remainders in reverse order. This process is repeated until you get a quotient of 0. To show that this works, let's pretend we're going to convert **153** from **Base 10** to **Base 10**. This is redundant but the method still works, and will show that it does work. Dividing **153** by **10** gives **15** with a remainder of **3**. Dividing **15** by **10** gives **1** with a remainder of **5**. Dividing **1** by **10** gives **0** with a remainder of **1**. Now that we have a quotient of **0**, we collect our remainders in the opposite order that we found them. This spells out the number **153**, which is what we expected since we didn't really convert bases. So let's try this where we want to convert **23** into **Binary**. Dividing by **2** over and over again will give the following equations.

- **23/2 = 11 r. 1**
- **11/2 = 5 r. 1**
- **5/2 = 2 r. 1**
- **2/2 = 1 r. 0**
- **1/2 = 0 r. 1**

Listing out the remainders backwards spells out **10111**, which is indeed **23** in **Binary**.

Now what if the number we're starting with is a fraction? Well rather than doing long division with decimals, we isolate the decimal portion. If we needed to convert **23.4** to **Binary**, we'd convert **23** by itself first to get **10111**. Now we convert **0.4**. Whereas previously we divided multiple times, now we'll multiply by the base we want. There are no remainders to collect, but instead we'll note the whole number portions of each result: that is the number to the left of the decimal. Then take the decimal portion of that number and repeat the process. We'll perform a few iterations of this below:

- **.4 * 2 = 0.8 -> 0**
- **.8 * 2 = 1.6 -> 1**
- **.6 * 2 = 1.2 -> 1**
- **.2 * 2 = 0.4 -> 0**
- **.4 * 2 = 0.8 -> 0**

We would continue this process until we got **0.0** as a product. Unfortunately, irrational numbers do exist, and so this process isn't guaranteed to terminate anytime soon. However, a handful of decimal places is plenty, especially if you recognize a pattern. Here we can notice that the last step was the same as the first, which means this method will repeat itself forever. So we can collect the first four integers that we noted and write them down in the order we collected them. Putting this together with our integer conversion from before gives us

$$23.4_{10} = 10111.011001100\ldots$$

and so on. Let's see another example of this. If we were asked to convert **2595.6875_{10}** to **Base 16**, we'd use the same approach: convert the integer and fractional parts of the number separately, and then add them back together.

Starting with the integer portion, we need to divide repeatedly by **16** because **16** is the base we want to convert to. Going through this process will look like this

- **2595/16 = 162 r. 3**
- **162/16 = 10 r. 2**
- **10/ 16 = 0 r.10 = A**

Notice how our last remainder was **10**. All of the remainders must be a single digit, so we'll write **10** as **A**. Collecting the remainders in reverse order gives us **A23**. Now we'll leave that for now and focus on the fractional piece. Remember we want to convert to **Base 16**, so we're going to multiply the fractional part by **16** repeatedly while collecting the numbers to the left of the decimal each time. Doing this will look like this:

- **0.6875*16 = 11.0 -> 11 = B**
- **0*16 = 0 -> Stop Here**

So the first time we multiplied by **16** we got **11** even. That means we note down the **11**, which is **B** in **Base 16**, and remove it from the number, giving us **0**. Because we now have **0**, our multiplication will only give **0**, so we can go ahead and stop here. We collect the whole numbers in the order we got them, so our fractional part will just be **B**.

Now we just add our two results together. So **2595.6875** in **Base 10** is **A23.B** is **Base 16**.

Base Conversions Not Involving Decimal

We won't always have the luxury of having **Base 10** as a starting point or end point. So what do we do in those situations? The methods we learned still work in other bases, but to use them would require us to learn arithmetic in different bases. The methods we learned take advantage of the fact that all the math we need to do works the same as what we're used to. So to avoid learning a new way to do arithmetic, we can use **Base 10** as a benchmark for our conversion: converting from our starting number to **Base 10**, and then taking that result and converting it to the desired base.

Imagine we were asked to convert **141** in **Base 11** to **Base 6**. If we knew how to do division in **Base 11** we could use our division method. But rather than doing that, we're going to convert to **Base 10** first. Using our polynomial expansion, we find that **141** is **166** in **Base 10**.

- $\mathbf{141_{11} = 1*11^2 + 4*11^1 + 1*11^0 = 121 + 44 + 1 = 166_{10}}$

Now we'll divide **166** by **6** over and over and collect the remainders backwards.

- **166/6 = 27 r.4**
- **27/6 = 4 r.3**
- **4/6 = 0 r.4**

So we find that **141** in **Base 11** is **434** in **Base 6**.

Arithmetic in a Different Base

Conversion isn't the only thing we can do with number systems. Often we need to arithmetic in these other bases. However, as we learned just now, we want to avoid arithmetic in new bases. So it's best to convert both operands to **Base 10**, do the math, and then convert back to whatever base you need your answer to be in. This seems like extra work, but it's actually faster and you'll make less mistakes.

Let's say we need to find **A+B** and **A-B** for **A = 54** and **B = 32**, both in **Base 7**. Let's first convert both of these to **Base 10**.

- $\mathbf{54_7 = 5*7 + 4 = 35 + 4 = 39_{10}}$
- $\mathbf{32_7 = 3*7 + 2 = 21 + 2 = 23_{10}}$

Now let's do **A+B** and **A-B**.

- **39 + 23 = 62**
- **39 – 23 = 16**

Now that we have our results, we need to convert them back into **Base 7** since that's where we started. Let's convert **62** first using our division method.

- **62/7 = 8 r.6**
- **8/7   = 1 r.1**
- **1/7   = 0 r.1**
- **62_{10} = 116_7**

And now let's convert **16**.

- **16/7 = 2 r.2**
- **2/7   = 0 r.2**
- **16_{10} = 22_7**

Alright so our final answer is **A+B** is **116** and **A-B** is **22** in **Base 7**.

Let's do one more example. Same operations, but this time with **B9** and **93** in **Base 16**. I encourage you to pause and try this yourself and then verify with my answers. First we need to convert to **Base 10**.

- **B9 = 11*16 + 9 = 185_{10}**
- **93 = 9*16 + 3 = 147_{10}**

Now let's do our addition and subtraction in **Base 10** like this.

- **185 + 147 = 332**
- **185 – 147 = 38**

Now convert the results back into **Base 16**.

- **332/16 = 20 r. 12**
- **20/16   = 1   r. 4**
- **1/16   = 0   r. 1**
- **332_{10} = $14C_{16}$**

- **38/16 = 2 r.6**
- **2/16   = 0 r.2**
- **38_{10} = 26_{16}**

Outro

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