# TutorTube: 2nd Derivative Test

Spring 2020

Hello and welcome to TutorTube. This video is going to cover 2nd derivative test applications to concavity and local extrema.

The second derivative test can be used to determine the concavity of a function, and if a certain stationary point in a function is a local extremum. By conducting the second derivative test, three properties of a function can be obtained:

- the local maximum
- the local minimum
- the inflection points

Before beginning the second derivative test, you must know and understand how to find the first derivative of a function and what the first derivative test is.

If you would like to see a review of the first-derivative test or practice derivative, please click on the link at the bottom of the screen to watch a video on it.

## Finding the Second Derivative of a Function.

The second derivative of a function is obtained the same way the first derivative is obtained, using basic derivative rules covered in Calculus 1. For example, let's find the second derivative of the following function f of x equals 5x square plus 3x plus 4

$$f(x) = 5x^2 + 3x + 4$$

From our knowledge of basic derivative rules, we would just simply use the power rule. We should also know automatically that the first derivative of this function, is f prime of x equals 10x plus 3

$$f'(x) = 10x + 3 \quad (1)$$

Now, we would apply the same derivative rules and techniques to equation 1 to find the second derivative of the function which is f double prime of x equals 10

$$f''(x) = 10 \quad (2)$$

This is how you would find the second derivative of a function.

Let's try a function with a bit more complex parts:

Let us find the derivative of f of x equals 3x to the fourth plus 5, all square, plus 2x square plus 6x.

$$f(x) = (3x^4 + 5) + 2x^2 + 6x$$

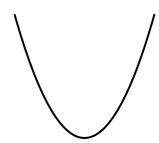
We would find the first derivative of the function, f'(x), which is 2 times 3x to the fourth plus 5 times 12x cube plus 4x plus 6. If you would notice, we have applied the chain rule to this first part, and we would do so again to find the second derivative. f double prime of x equals 2 times 12x cube times 12x cube, and I obtained these by applying the product rule, plus 36x square times 3x to the fourth plus 5, all plus four. And if we simplify this entire thing, we would end up having 2 times 152x to the 6th plus 180x square plus 4, and that would further simplify to f double prime of x equals 304x to the 6th plus 360x square plus four and that is our second derivative.

We would use this knowledge to find the second derivatives of other functions when carrying out concavity and extrema tests.

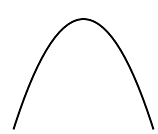
First, we will start with concavity test:

## Concavity Test

Concavity can be defined as the rate at which a function's derivative changes. This is usually referred to as concave up (opening upward as a normal U),



or concave down (opening downward like an up-side-down U).



The concavity theorem states that if f double prime is greater than zero on N, then f is concave up on that open interval N if f double prime is less than zero on N, then f is concave down on that open interval N however, if there is any point *c*, on the interval, where f'' changes its sign (+ve to −ve or vice versa), then that point *c*, is an inflection point of f

## Theorem 4.6:

If a certain f'' exists on an open interval N, then we can use the following to determine the concavity of the function:

- if f'' > 0 on N, then f is concave up on that open interval N
- if f'' < 0 on N, then f is concave down on that open interval N
- if there is any point *c*, on the interval, where f'' changes its sign (+ve to −ve or vice versa), then that point *c*, is an inflection point of f

Let's try a couple examples to get you familiar with this.

Let us determine the concavity and find the inflection points of f(x) = $3x^4 - 4x^3 - 6x^2 + 12x + 1$.

$f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$

Find the first derivative of the function, f'.

$f'(x) = 12x^3 - 12x^2 - 12x + 12$

Find the second derivative of the function, f''.

$f''(x) = 36x^2 - 24x - 12$

Next, find the inflection points by setting f'' equal to 0 and solving for x.

$f''(x) = 36x^2 - 24x - 12$
$0 = 36x^2 - 24x - 12$
$0 = 12(3x^2 - 2x - 1)$
$0 = 12(3x - 1)(x - 1)$
$x = -\frac{1}{3}, 1$

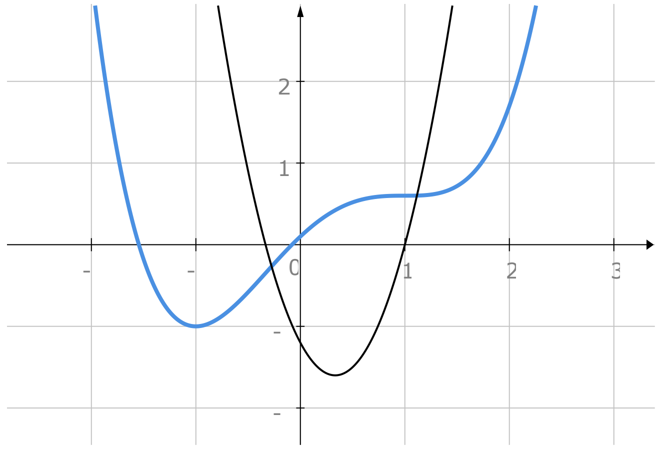
Now, check the concavity of the function. Do this by checking for sign changes of f'' at different points. Each change in sign determines the behavior of f. For each point, we make reference to theorem 4.6 to determine the concavity.

We can create a little table to contain the data like this:

$-1/3$ $\quad$ $1$

| | -1 | 0 | 3 |
|---|---|---|---|
| f'' | 48 | -12 | 240 |
| f | ∪ | ∩ | ∪ |

We can now see that on $\left(-\infty, -\frac{1}{3}\right)$ and $(1, \infty)$, the function is concave up, and concave down on $\left(-\frac{1}{3}, 1\right)$.

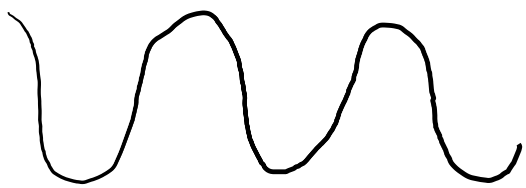



This rounds up the concavity test. We will now work on the local extrema test.

## Local Extrema

Now, the second derivative test, as mentioned before, is also used to identify local maxima and minima of a function.

Assuming we have a function, and the graph of that function looks like this:



After finding the second derivative (f''), the next step would be to confirm that f'' is continuous on an open interval where $f'(c) = 0$. The following can then be observed:

- If $f''(c) > 0$, f has a local minimum at c
- If $f''(c) < 0$, f has a local maximum at c
- If $f''(c) = 0$, the test is inconclusive

In this context, inconclusive means the graph either has a local maximum, minimum, or neither.

Now, let us try our first two examples.

a. Let us use the second derivative test to find the local extrema of $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$ on the interval [-2,2].

First, we find the first derivative of the function which is $f'(x) = 12x^3 - 12x^2 - 12x + 12$.

After finding the first derivative of the function, we have to find the critical points using the first derivative. Do this by setting f'(x) = 0, and solve for all x values:

$$f'(x) = 0$$
$$0 = 12x^3 - 12x^2 - 12x + 12$$
$$12 = -12x^3 + 12x^2 + 12x$$

Divide through by 12 to get

$$1 = -x^3 + x^2 + x$$

Factor out x from the right-hand side of the equation

$$1 = x(-x^2 + x + 1)$$

Factorize the portion in the parenthesis

$$x = -1, 1$$

Hence, we have our critical points as -1 and 1.

Next, we find the second derivative of the function, which is f''(x) = $36x^2$ - 24x - 12.

We then make reference to the observations listed previously, to confirm what type of local extremum is contained in the function:

- Plug in each of the critical points to the second derivative of the function
- if the resulting value is greater than 0, then there is a local minimum at that critical point
- if the resulting value is less than 0, then there is a local maximum at that critical point
- if the resulting value is 0, then the test is inconclusive

$$f''(-1) = 36(-1)^2 - 24(-1)\text{-}12$$
$$f''(-1) = 48$$

This test shows that there is a local minimum at x = -1.

$$f''(1) = 36(1)^2 - 24(1) - 12$$
$$f''(1) = 0$$

This test is inconclusive since a value of 0 was obtained. Hence, the function does not have a local maximum or minimum at this point.

## Outro

Alright, this rounds up our video. Don't forget to visit our YouTube channel for more videos or visit our website to explore and use more of the UNT Learning Center's resources.