TutorTube: Related Rates

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Introduction

Hello! Welcome to TutorTube, where The Learning Center's Lead Tutors help you understand challenging course concepts with easy to understand videos. My name is Ebby, Lead Tutor for Math and Political Science. In today's video, we will explore Related Rates. Let's get started!

Example I

The top of a ladder slides down a vertical wall at a rate of **0.25 m/s**. At the moment when the bottom of the ladder is **5m** from the wall, it slides away from the wall at a rate of **0.15 m/s.** How long is the ladder?

What do we know about right triangles? They are governed by the Pythagorean Theorem:
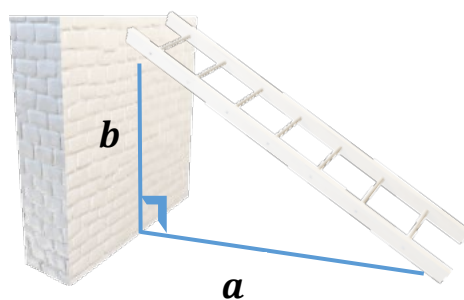
$$a^2 + b^2 = c^2.$$

What do we know about Related Rates? We know we need to take the derivative of the equation that we are given. So, this becomes
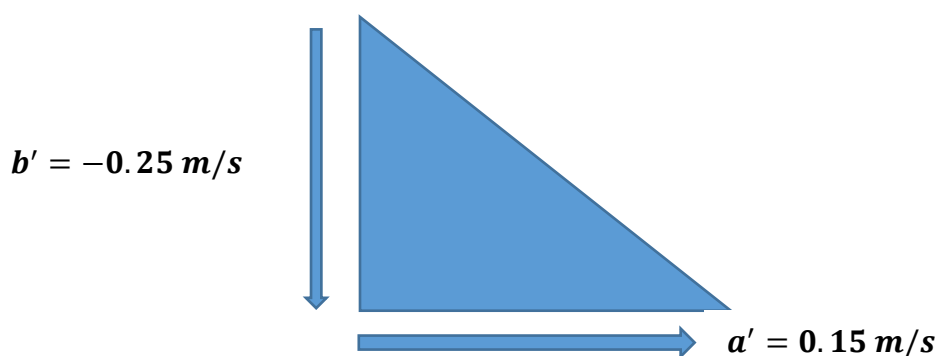
$$2a \cdot a' + 2b \cdot b' = 2c \cdot c'.$$

Notice that you can divide everything by 2 and you're left with

$$a \cdot a' + b \cdot b' = c \cdot c'.$$

All that's left to do now is to assign variables,

Now that we have **a** and **b,** we can also get $a'$and b'. From the problem, we know that:

So, that leaves us with **c** and **c′**. We know that c' represents the rate at which the ladder changes length. But since the ladder never changes length (that would be bad) c' would be 0. Since we know this, we can further simplify this to be

$$0.75 - 0.25b = 0$$

So, we solve for **b**, which is how long the wall is, and get **3m.**

$$b = 3m$$

$$a = 5m$$
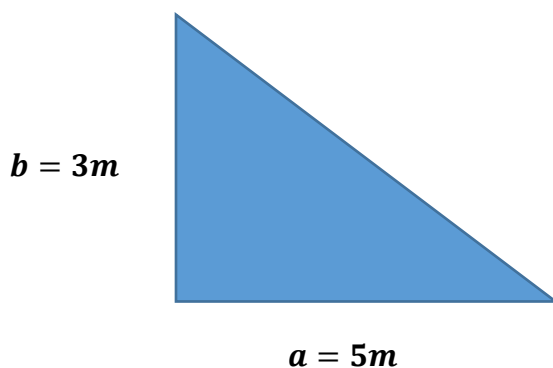
So, how does that help us find the length of the ladder? We go back to the Pythagorean Theorem. Since we have **a** given as **5m** and we have **b** given as **3m**, we get

$$5^2 + 3^2 = c^2.$$

Utilizing algebra to solve for **c**, we get

$$\sqrt{34} = 5.8.$$

So, the ladder is **5.8 meters long**.

Example II

In This second example here: water is leaking out of a tank that is shaped like an inverted circular cone at a rate of **10,0000** $\frac{cm}{min}$. At the same time, water is being pumped into the tank at a constant rate. The base diameter of this tank is **4m**

and the height is **8m**. If the water is rising at a rate of **15** $\frac{cm}{min}$ when it is **2m** deep, find the rate at which the water level is being pumped into the tank.

So, this is a differential equations problem, but if we solve for one of these variables, we can make it a related rates problem. But the first thing we'll do is make sure that everything is in the same units. Since the units of the rate that is given is in centimeters, we want to change everything to centimeters. So, the diameter of the base is **400cm**. Since the radius is the diameter divided by 2, it would equal **200cm**. We want radius because that's what we use in the volume equation, and the height of the entire cone itself is **800cm** (Stewart, 249).

$$r = 200cm$$

$$h = 800cm$$
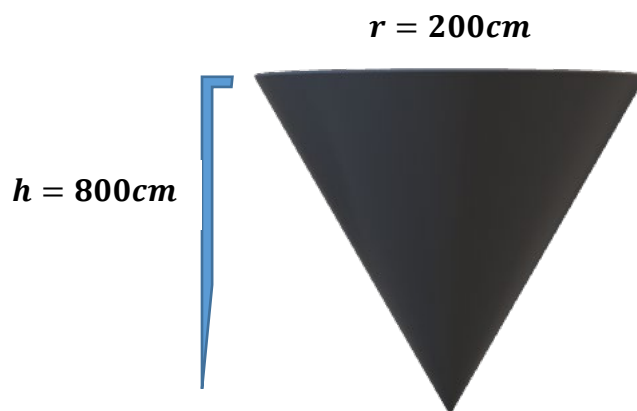
The question asks us to figure out the rate at which the water is being pumped into the tank. So, how we express that is by using the formula for volume change, which is denoted as v'. In this problem, volume change represents the difference between how much water is being poured in and how much is poured out. So, we will call the rate at which water is being pumped in p' and determine our volume change equation to be

$$v' = p' - 10,000 \; \frac{cm^3}{min}.$$

Next, remember that the radius is **200cm** and the height of the entire tank itself is **800cm**. So, using those two things, we know that

$$r = \frac{1}{4}h.$$

So how is that helpful? We use it to substitute into our volume equation:

$$v = \frac{1}{3}\pi r^2 h.$$

So, it becomes

$$v = \frac{1}{3}\pi \cdot \left(\frac{1}{4}h\right)^2 \cdot h.$$

Why exactly do we use **h** instead of **r** though? Since we have **h'** as **15 $\frac{cm}{min}$** we need to get everything in terms of **h** in order to plug in **h'** when we take the derivative. But first, we need to simplify the **v** expression into

$$v = \frac{\pi}{48}h^3.$$

Then we can take the derivative in order to bring about **h'**:

$$v' = \frac{\pi}{16}h^2 \cdot h'$$

So, now that we have **v'** we can substitute it into

$$v' = p' - 10,000 \; \frac{cm^3}{min}$$

and get

$$\frac{\pi}{16}h^2 \cdot h' = p' - 10,000 \; \frac{cm^3}{min}.$$

Once we do that, we also need to plug in **h'**, the rate at which the water level is rising, and how deep it is when that rate is rising. We know it is **200cm** deep and it's going **15 $\frac{cm}{min}$** when it's at that depth, therefore we get

$$\frac{\pi}{16}(200)^2(15) = p' - 10,000.$$

All we need to do now is solve for **p'** and once we do that, we get

$$37,500\pi + 10,000 = p'.$$

So, **p'** is

$$127,810\frac{cm^3}{min}.$$

Example III

You are situated **16 km** from the launching pad of a rocket. Suppose the rocket travels vertically at **1200$\frac{km}{h}$**. Find the rate at which the angle θ, between the telescope and the ground, is increasing **3 minutes (0.05 hours)** after liftoff. So, the rocket is traveling **1200$\frac{km}{h}$** straight up, and in this problem, we will not be worrying about any drag or air resistance, we're going to keep it as simple as possible.
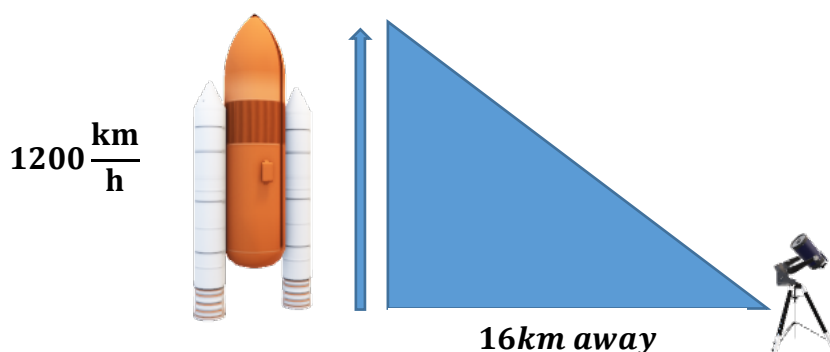
We are situated **16km** away from the rocket and we want to figure out the rate at which the angle θ, between the telescope and the ground is increasing at a particular time. Notice that this is a right triangle, but instead of Pythagorean Theorem we will be using trig. Specifically,

$$\tan \theta = \frac{y}{x}.$$

In this case,

$$y = 1200\frac{\text{km}}{\text{h}} \text{ and } x = 16\text{km}.$$

Remember that these are different units, so when we do that division, we get

$$\frac{75}{h}.$$

and the kilometers cancel. So, we have **h** on the denominator and the expression can be simplified to

Remember that **tan θ** is supposed to be unitless and we need to cancel out the hours. So, we will multiply by the variable **t** hours.

$$\tan \theta = \frac{75}{h} \cdot (t)h$$

t is in hours and it cancels out with the hours on the denominator yielding:

$$\tan \theta = 75(t).$$

Remember that we need to solve for θ', and we know we can get the actual angle between the telescope and the ground 3 minutes after liftoff. So, we substitute

$$\tan \theta = 75(0.05) \rightarrow \frac{15}{4}.$$

So, how do we get θ? We use tangent inverse:

$$\theta = \tan^{-1}\left(\frac{15}{4}\right).$$

Now, in order to solve the problem, we want θ' and we've gotten θ to help us. Let's use the expression that we found,

$$\tan\theta = 75(t)$$

and take the derivative of that. Taking the derivative of the left and right-hand side gives:

$$\sec^2(\theta) \cdot \theta' = 75$$

Remember that $\sec^2(\theta)$ is really $\frac{1}{[\cos\theta]^2}$, so we may write:

$$\frac{1}{[\cos\theta]^2} \cdot \theta' = 75.$$

We now have an expression that includes θ' as well as θ. We will utilize algebra to isolate θ' to be:

$$\theta' = 75[\cos\theta]^2.$$

Then we will plug the θ value into the expression and obtain:

$$75\cos\left(\tan^{-1}\left(\tfrac{15}{4}\right)\right)^2|$$

It is better to keep everything exact, so I would advise not to plug in a rounded θ value. Plugging this into the calculator, you will get

$$\theta' = 4.98$$

rounding off to two decimal places, and that is in **radians per hour**.

Outro

Thank you for watching TutorTube! I hope you enjoyed this video. Please subscribe to our channel @UNTLC for more exciting videos! Check out the links in the description below for more information about The Learning Center and follow us on social media. See you next time!

References

Stewart, James. *Calculus: Early Transcendentals*, 7th Ed., Brooks/Cole Cengage Learning, 2010