TutorTube: Derivatives and Their Applications

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Introduction

Hello! Welcome to TutorTube, where The Learning Center's Lead Tutors help you understand challenging course concepts with easy to understand videos. My name is Ebby, Lead Tutor for Math and Political Science. In today's video, we will explore Derivatives and Their Applications. Let's get started!

Power Rule

Power rule is written as this:

$$\frac{d}{dx}cx^n = cnx^{n-1},$$

where we bring down the power, multiply it by the coefficient that's already there, and subtract the old power by 1. To illustrate how it works, let's solve the following examples:

1. x^2
2. $3x^3 - x + 13$
3. $x^8 + 2x^4 + 8x$

Applying power rule to number one yields

$$2x^{2-1} = \mathbf{2x}.$$

Applying power rule to number two yields

$$3 \cdot 3x^2 - 1x^0 = \boldsymbol{9x^2 - 1}.$$

Lastly, applying power rule to number 3 yields

$$8 \cdot x^7 + 4 \cdot 2x^3 + 8 = \boldsymbol{8x^7 + 8x^3 + 8}.$$

Chain Rule

Chain rule is written as this

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x),$$

where we identify an inner and outer function, find the derivative of the outer, then multiply it by the derivative of the inner.

The following three examples will illustrate how to apply the chain rule:

1. $y = \sin(5x^2)\, h(x) = (x^2 + 3x - 1)^6$
2. f(t) = $\sqrt{1 - \sec 2t}$.

For number one, the outer function is sin and the inner function is $5x^2$, therefore we apply chain rule as such:

$$\cos(5x^2) \cdot [5x^2]' ,$$

simplifying the derivative of the inner we get,

$$\cos(5x^2) \cdot 10x$$

which can be re-arranged to

$$\mathbf{10x\cos(5x^2)} .$$

Example 2: first re-write it as

$$(1 - \sec 2t)^{\frac{1}{2}} .$$

Now, apply the chain rule:

$$\frac{1}{2}(1 - \sec 2t)^{-\frac{1}{2}} \cdot (1 - \sec 2t)' .$$

In this case, taking the derivative of the outer calls for re-applying the Chain Rule which yields

$$\frac{1}{2}(1 - \sec 2t)^{-\frac{1}{2}} \cdot - 2\text{secxtanx} ,$$

which can be simplified to

$$-\text{secxtanx}(1 - \sec 2t)^{-\frac{1}{2}} .$$

Finally, we will re-write the function with the radical:

$$\frac{\mathbf{-secxtanx}}{\sqrt{\mathbf{1 - sec\,2}t}} .$$

Product Rule

The Product Rule is used when taking the derivative of expressions that are being multiplied. It is written as

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

where we take the derivative of one expression, multiply it times the other, and add the opposite.

The following two examples will help us get a better understanding of how to apply it:

1. $(x^2-1)^4(x+5)$
2. $(x+3)\, ln(x+3)$

The procedure for applying the Product Rule in the first example is as follows:

we will take the derivative of the $(x^2-1)^4$ first, which means that we will multiply by $(x+5)$ and add the opposite:

$$[(x^2-1)^4]'(x+5)+(x^2-1)^4(x+5)'\,.$$

Now we must apply the derivatives

$$4(x^2-1)^3(2x)(x+5)+\ (x^2-1)^4(1),$$

remember that the derivative of $(x^2-1)^4$ requires chain rule, so we need to also multiply by 2x. Next, we will simplify by factoring out a common $(x^2-1)^3$:

$$(x^2-1)^3[4(2x)(x+5)+(x^2-1)^1(1)]\,,$$

which can be simplified to

$$(x^2-1)^3[(8x^2+40x)+(x^2-1)]\,,$$

and finally:

$$\boldsymbol{(x^2-1)^3[9x^2+40x-1].}$$

We will follow the same procedure for example two:

$$(x+3)'\ ln(x+3)+\ (x+3)(ln(x+3))'.$$

Since the derivative of the natural log is 1 over the argument (inner function) multiplied by the derivative of the argument, we may write

$$(1)\, ln(x+3)+(\mathrm{x}+3)\left(\frac{1}{x+3}(x+3)'\right).$$

Now we simplify the expression:

$$ln(x+3)+(\mathrm{x}+3)\left(\frac{1}{x+3}(1)\right),$$

which gives us

$$\boldsymbol{ln(x+3)+1}\,.$$

Quotient Rule:

The Quotient Rule is written as

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Let's work through two examples:

1. h(x) = $\frac{x^2-4x-3}{x+3}$

2. t(x) = $\frac{5x^3+6x}{5-x^3}$

in order to get a better understanding on how to apply it. Quotient rule requires us to first take the derivative of the numerator, multiply it times the denominator, and **subtract** the opposite, which will become the new numerator:

$$(x^2 - 4x - 3)'(x + 3) - (x^2 - 4x - 3)(x + 3).$$

Then, we must square the original denominator and put everything together into a new fraction as such

$$\frac{(x^2 - 4x - 3)'(x + 3) - (x^2 - 4x - 3)(x + 3)'}{(x + 3)^2}.$$

Now, we simplify the numerator

$$\frac{(2x - 4)(x + 3) - (x^2 - 4x - 3)(1)}{(x + 3)^2},$$

Distribute the negative

$$\frac{(2x^2 + 6x - 4x - 12) - x^2 + 4x + 3}{(x + 3)^2},$$

and simplify again

$$\boldsymbol{\frac{x^2 + 6x - 9}{(x + 3)^2}}$$

$$\boldsymbol{\frac{(x - 3)^2}{(x + 3)^2}}.$$

Applying the same principles to number two yields

$$\frac{(5x^3 + 6x)'(5 - x^3) - (5x^3 + 6x)(5 - x^3)'}{(5 - x^3)^2},$$

which simplifies to

$$\frac{(15x^2+6)(5-x^3)-(5x^3+6x)(-3x^2)}{(5-x^3)^2}.$$

now distribute (check out our TutorTube if you need a quick re-fresher on Distributing/Foiling)

$$\frac{75x^2-15x^5+30-6x^3-(-15x^5-18x^3)}{(5-x^3)^2}$$

$$\frac{75x^2-15x^5+30-6x^3+15x^5+18x^3}{(5-x^3)^2},$$

and simplify

$$\boldsymbol{\frac{12x^3+75x^2+30}{(5-x^3)^2}}.$$

Implicit Differentiation

Finding the derivative implicitly requires us to derive and isolate y' ourselves. Let's take a look at the following examples:

$$1.\ \sin(x+y)=y^3\cos x$$

$$2.\ \frac{x^2}{x+y}=y^3+2,$$

and practice how to find y'. Since y is located on both sides of the equation, we will take the derivative of both sides **with respect to x (meaning that we're taking derivatives of the variables with x's):**

$$(x+y)'cos(\mathrm{x}+\mathrm{y})=(y^3)'\cos(\mathrm{x})+(y^3)\big(\cos(\mathrm{x})\big)'.$$

Since we want to derive y', **we include it every time that we take the *derivative* of y;** so, we have:

$$(1+y')cos(\mathrm{x}+\mathrm{y})=(3y^2y')\cos(\mathrm{x})+(y^3)\big(-\sin(\mathrm{x})\big).$$

Now, our goal is to group everything with a y' together and isolate them, so we will first distribute:

$$cos(\mathrm{x}+\mathrm{y})+y'cos(\mathrm{x}+\mathrm{y})=3y^2\mathrm{y}'\cos(\mathrm{x})+(y^3)\big(-\sin(\mathrm{x})\big),$$

then **group the variables with y' together**

$$y'cos(\mathrm{x}+\mathrm{y})-3y^2\mathrm{y}'\cos(\mathrm{x})=y^3\big(-\sin(\mathrm{x})\big)+cos(\mathrm{x}+\mathrm{y}),$$

then **factor out y'**

$$y'[\cos(x+y) - 3y^2\cos(x)] = -y^3\sin(\mathrm{x}) + \cos(\mathrm{x}+\mathrm{y})\,,$$

and finally, **isolate y'**

$$\boldsymbol{y'} = \frac{\boldsymbol{-y^3\sin(\mathrm{x}) + \cos(\mathrm{x}+\mathrm{y})}}{\boldsymbol{\cos(x+y) - 3y^2\cos(x)}}.$$

We will apply the same principles for number two: first take the derivative of both sides, which yields

$$\frac{(x^2)'(x+y) + (x^2)(x+y)'}{(x+y)^2} = (y^3)' + 0.$$

Make sure to **multiply all derivatives of y with y':**

$$\frac{2x(x+y) + (x^2)(1+y')}{(x+y)^2} = 3y^2y'.$$

Now, clear the fraction by multiplying it on both sides

$$2x(x+y) + (x^2)(1+y') = 3y^2y'(x+y)^2$$

then work to **group the variables with y' and isolate y':**

$$-3y^2y'(x+y)^2 + (x^2)(1+y') = -2x(x+y)$$

then distribute:

$$-3y^2y'(x+y)^2 + x^2 + x^2y' = -2x(x+y)\,,$$

Group all variables with y' on one side

$$-3y^2y'(x+y)^2 + x^2y' = -2x(x+y) - x^2\,,$$

Factor out y'

$$y'[-3y^2(x+y)^2 + x^2] = -2x(x+y) - x^2\,,$$

and finally, isolate y'

$$\boldsymbol{y'} = \frac{\boldsymbol{-2\mathrm{x}(\mathrm{x}+\mathrm{y})} - x^2}{\boldsymbol{-3y^2(x+y)^2 + x^2}}$$

$$\boldsymbol{y'} = \frac{\boldsymbol{-3x^2 - 2xy}}{\boldsymbol{-3y^2(x+y)^2 + x^2}}.$$

First and Second Derivative Test

For the following function

$$f(x) = \sin\frac{x}{2} + 1 \text{ on the interval } (0,2\pi).$$

we wish to determine whether there exists a maximum and minimum (and where) as well as the interval(s) of concavity. To do this, we will apply the First and Second Derivative Test. First, we will find any maximum(s) or minimum(s) using the First Derivative Test, which involves taking the derivative:

$$f'(x) = \cos(\tfrac{x}{2}) \cdot \tfrac{1}{2},$$

setting it equal to 0

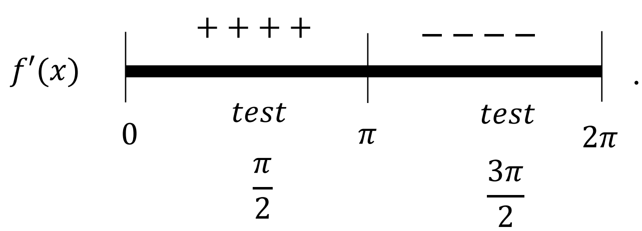
$$\tfrac{1}{2}\cos(\tfrac{x}{2}) = 0\,,$$

and solving

$$\frac{x}{2} = \frac{\pi}{2}, \frac{3\pi}{2} \text{ on } \left(\frac{0}{2}, \frac{2\pi}{2}\right)$$

$$x = \pi, 3\pi \text{ on } [0, \pi] \rightarrow \boldsymbol{x = \pi}\,.$$

Note that the argument of cos is $\frac{x}{2}$, so we must adjust the domain accordingly. Solving the derivative equation yielded $\boldsymbol{\pi}$ **as the critical value** which means that we must test whether or not there is a maximum or a minimum there:



Since the function is increasing (has a positive slope) prior to the critical value at x=π, and decreasing (negative slope) afterwards, we know that there is ***a maximum* at x= $\boldsymbol{\pi}$**. Since there is only one critical value in the given interval ***and* it's a maximum**, we know that there is no minimum (since the endpoints aren't inclusive). We also want to determine where the maximum is located on the graph, to do this, we must plug the critical value into the original function:

$$f(\pi) = \sin\frac{\pi}{2} + 1 \rightarrow \mathbf{2}\,.$$

Therefore, the coordinates for the maximum over the given interval are:

$$(\boldsymbol{\pi}, \mathbf{2})\,.$$

Now, we wish to determine the interval(s) of concavity. To do this, we must apply the second derivative which yields:

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right),$$

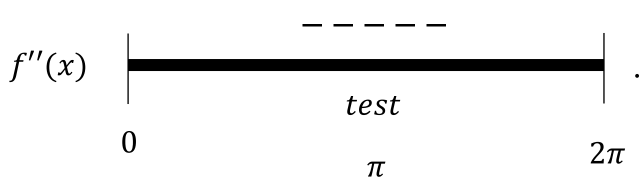
set it equal to 0 and solve

$$-\frac{1}{4}\sin\left(\frac{x}{2}\right) = 0$$

$$\frac{x}{2} = 0, \pi, 2\pi \; on \; [0, \pi]$$

$$\frac{x}{2} = 0, \pi \rightarrow \mathbf{0},\ \mathbf{2\pi}\ .$$

Now we must assess the interval(s) of concavity:



Since the second derivative is **decreasing** over the entire interval (negative slope) we know that the function is **concave down and never concave up** over the given interval.

Applications of Derivatives

We will examine two common applications of derivatives:

1. Finding the equation of the tangent line
2. Determining where the tangent line is horizontal

To illustrate the first application, we will examine the following problem:

Find the equation of the tangent line to the curve $y = x + tan(x) \; at \; x = \pi$.

To solve this, ***we will find the derivative and determine the slope, then use the point slope formula.***

$$y' = 1 + sec^2x$$

$$= 1 + \frac{1}{cos^2x}$$

The point slope formula is

$$y - y_1 = m(x - x_1)$$

where we must determine x_1 , y_1 , and m . We can find x_1 , y_1 from the given information:

$$x_1 = \boldsymbol{\pi}$$

because the problem specified "at x = $\boldsymbol{\pi}$"

and

$$y_1 = \pi + 0 \ \rightarrow \boldsymbol{\pi}$$

From plugging in $\boldsymbol{\pi}$ into the given equation. To determine m though, we must plug in x_1 into the derivative equation that we found earlier:

$$m = 1 + \frac{1}{cos^2(\pi)} \rightarrow \mathbf{2} \, .$$

Now that we have all the missing variables, we can plug each of them into the point slope formula and find the equation of the tangent line

$$y - \pi = 2(x - \boldsymbol{\pi})$$

$$\boldsymbol{y = 2x - \pi} \, .$$

Our second example is as follows:

Find the points on the curve $y = 2x^3 - 4x^2 - 14x + 10$ where the tangent line is horizontal.

To do this, we will **set the derivative function equal to 0 and simplify:**

$$y' = 6x^2 - 8x - 14$$

$$0 = 6x^2 - 8x - 14$$

$$0 = 2(3x^2 - 4x - 7)$$

Now factor (TutorTube) and solve:

$$0 = 2(3x - 7)(x + 1)$$

$$x = \frac{7}{3} \, , \ -1 \, .$$

To determine the coordinates, we must plug each individual critical value into the original function:

$$f(-1) = 18$$

$$f\left(\frac{7}{3}\right) = -\frac{514}{27}$$

Therefore, the following points:

$$(-\mathbf{1}, \mathbf{18}), (\frac{\mathbf{7}}{\mathbf{3}}, -\frac{\mathbf{514}}{\mathbf{27}})$$

correspond to where the tangent line is horizontal.

Outro

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