TutorTube: Further Applications of Derivatives

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Introduction

Hello! Welcome to TutorTube, where The Learning Center's Lead Tutors help you understand challenging course concepts with easy to understand videos. My name is Ebby, Lead Tutor for Math and Political Science. In today's video, we will explore Further Applications of Derivatives. Let's get started!

Challenging Derivatives

Let's take a look at some more challenging derivatives:

1. y = $\sqrt[3]{1+cot^2x}$
2. y =$cos^4(sin^3 2x)$

For number 1 we must change it into exponential:

$$(1+cot^2x)^{\frac{1}{3}}.$$

Then we will apply the chain rule:

$$\frac{1}{3}(1+cot^2x)^{-\frac{2}{3}} \cdot (1+cot^2x)'.$$

We simplify this by doing chain rule again for $1+cot^2x$, which yields:

$$\frac{1}{3}(1+cot^2x)^{-\frac{2}{3}} \cdot (2cotx) \cdot -csc^2x.$$

Now we simplify:

$$\frac{1}{3}(1+cot^2x)^{-\frac{2}{3}} \cdot -2csc^2xcotx$$

Put the negative exponent on the denominator

$$\frac{-2csc^2xcotx}{3\,(1+cot^2x)^{\frac{2}{3}}}$$

and put it back into a root:

$$\boldsymbol{\frac{-2csc^2xcotx}{3\sqrt[3]{(1+cot^2x)^2}}}.$$

For number two we must again apply the Chain Rule first. To make it easier, let's call $sin^3 2x$ 'a':

$$[\cos(a)]^4$$

Now apply the Chain Rule:

$$4[\cos(a)]^3 \cdot (\cos(\mathrm{a}))',$$

Apply it again to the cos(a)

$$4[\cos(a)]^3 \cdot -sin(\mathrm{a}) \cdot \mathrm{a}'.$$

Now substitute 'a' with $sin^3 2x$

$$4[\cos(sin^3 2x\ )]^3 \cdot -sin(sin^3 2x) \cdot (sin^3 2x)'.$$

Notice that we have to take the derivative of the last $sin^3 2x$ (at the end):

$$4[cos(sin^3 2x)]^3 \cdot -sin(sin^3 2x) \cdot (3sin^2 2x) \cdot cos(2x) \cdot 2$$

Now simplify by putting the numbers in front and the trig values afterwards.

$$\boldsymbol{-24cos^3(sin^3 2x)\, cos(2x) sin^2(2x) \sin(sin^2 2x)}$$

More Challenging Derivatives

Let's continue with more challenging derivatives by evaluating the following functions:

1. $y = 3 \cdot 4^{7x^2+5}$
2. $\ln \sqrt[3]{\frac{x^2+5x+2}{x+1}}$

Since the first expression has an x in the exponent, we cannot use power rule to take the derivative. Therefore, we must move the x from the exponent by taking the natural log of the function

$$\ln y = \ln(3 \cdot 4^{7x^2+5}),$$

and using Log Rules to simplify:

$$\ln y = \ln 3 + \ln(4^{7x^2+5})$$

$$\ln y = \ln 3 + (7x^2 + 5) \ln 4 .$$

Now that we have the x out of the exponent, we can distribute the ln 4 like this:

$$\ln y = \ln 3 + 7x^2 \ln 4 + 5 \ln 4 ,$$

and now take the derivative of both sides using implicit differentiation. When taking the derivative of the left side we must include the y' and on everything on the right side that has an ln yields 0:

$$\frac{1}{y} \cdot y' = 0 + 14xln4 + 0(7x^2) + 0\,.$$

Then simplify and solve for y'

$$y' = y[14xln4]$$

and **substitute the original y function back into y'**

$$y' = 3 \cdot 4^{7x^2+5}[14xln4]$$

We can't simplify the 4 because it is inside of ln, but we can simplify the 14x with the 3:

$$\boldsymbol{y' = (42xln4)4^{7x^2+5}}\,.$$

Now, let's examine the second function. Similar to the first problem, **we must use algebra/pre-calculus to simplify the function before taking the derivative.** In general, simplifying the given function first will often reduce the complexity of taking its derivative. The first step is to change the exponent into a fraction:

$$\ln\left(\frac{x^2 + 5x + 2}{x + 1}\right)^{\frac{1}{3}}\,,$$

then bring the exponent down and expand the log

$$\frac{1}{3}\ln\frac{x^2 + 5x + 2}{x + 1}\,.$$

Since the log is being divided on the inside, we split it up using subtraction

$$\frac{1}{3}\ln x^2 + 5x + 2 - \frac{1}{3}\ln x + 1\,.$$

Now that we have sufficiently simplified the function, we are ready to take the derivative. In this case we will **use Chain Rule to find the derivative of both natural logs**, applying it yields

$$\frac{1}{3}\left(\frac{1}{x^2 + 5x + 2}\right) \cdot (2x + 5) - \frac{1}{3}\left(\frac{1}{x + 1}\right) \cdot 1$$

$$\frac{1}{3}\left[\left(\frac{2\text{x} + 5}{x^2 + 5x + 2}\right) - \frac{1}{x + 1}\right]$$

which we can simplify by getting common denominators

$$\frac{1}{3}\left[\frac{2\mathrm{x}+5(\mathrm{x}+1)-x^2-5x-2}{(x+1)(x^2+5x+2)}\right]$$

and distributing the top

$$\frac{1}{3}\left[\frac{2\mathrm{x}^2+7x+5-x^2-5x-2}{(x+1)(x^2+5x+2)}\right]$$

$$\boldsymbol{\frac{x^2+2x+3}{3(x+1)(x^2+5x+2)}}.$$

Application Questions

Let's take a look at some application questions. The first is:

What Is the average rate of change of the function $\sqrt{2x+1}$ from 0 to 5?

To answer this kind of question, we must **know the average rate of change formula**, which is expressed as this:

$$\frac{f(b)-f(a)}{b-a};\ \ b-a\ \neq 0\,.$$

We can informally characterize this as the distance "we went" subtracted by the point "we started at" all over the time it took. Then plug in the given values into the formula:

$$\frac{f(5)-f(0)}{5-0}.$$

Which yields

$$\boldsymbol{\frac{\sqrt{11}-1}{5}.\ About\ 0.463\ \frac{units}{unit}}.$$

For number two:

Suppose an object moves along a path (in feet) defined by the following function: $300x-20x^2$, where x is in seconds. **What Is the velocity between 0 and 3 seconds? What is the instantaneous velocity function? At what time does the object have no movement?**

To find the velocity, we will use the average velocity formula:

$$\frac{f(3)-f(0)}{3-0}$$

which yields

$$\frac{300(3) - 20(3)^2 - 0}{3}$$

$$\mathbf{240}\ {}^{\boldsymbol{ft}}/_{\boldsymbol{s}}\,.$$

To answer the second part of the question we must recognize that the **instantaneous velocity can be found by taking the derivative of the given function.** Therefore, we will take the derivative of

$$300x - 20x^2\,,$$

which is

$$\mathbf{300 - 40}\boldsymbol{x}\,.$$

The final part of the question wants us to determine at what particular time the object has no movement. This is **when the instantaneous velocity equals 0,** so we will set the derivative equal to 0

$$300 - 40x = 0$$

and solve

$$300 = 40x$$

$$\frac{15}{2} = x$$

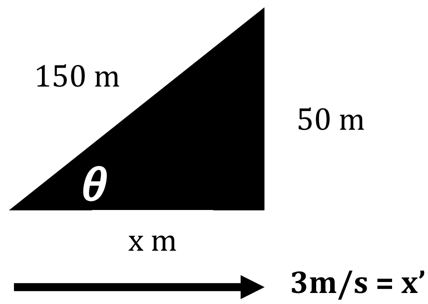
$$\mathbf{7.5}\ \boldsymbol{seconds}\,.$$

Related Rates

Let's now take a look at some Related Rates problems, we have a video dedicated to Related Rates, but we weren't able to cover every single topic that you may see, so here are two more examples:

1. A kite 50 m above the ground moves horizontally at a speed of 3 m/s. At what rate is the angle between the string and the horizontal decreasing when 150 m of string has been let out?
2. A streetlight is mounted at the top of a 20-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 8 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

To solve the first problem, we will create the following model



We can find the value of theta itself since we have the opposite and hypotenuse:

$$sin(\theta) = \frac{50}{150} = \frac{1}{3}$$

$$\theta = \sin^{-1}\frac{1}{3}.$$

We need to create a relationship that has θ and x' in it:

$$Cot(\theta) = \frac{x}{50}$$

taking the derivative of this will give us that relationship:

$$-\csc^2(\theta) \cdot \theta' = \frac{1}{50} \cdot x'.$$

Since we know theta and x' we substitute them into the equation:

$$-\csc^2\left(\sin^{-1}\frac{1}{3}\right) \cdot \theta' = \frac{1}{50} \cdot 3$$

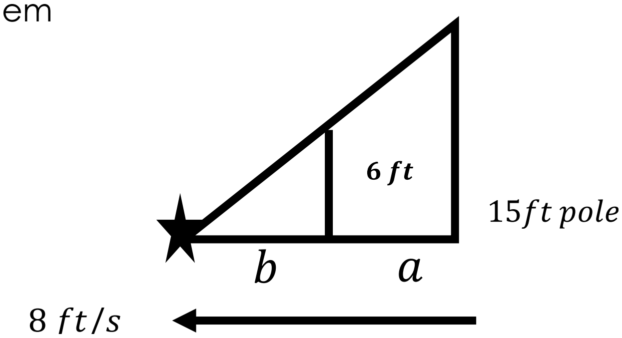
$$\frac{-1}{\sin^2(\sin^{-1}\frac{1}{3})} \cdot \theta' = \frac{3}{50},$$

then solve for θ' (the rate at which the angle changes):

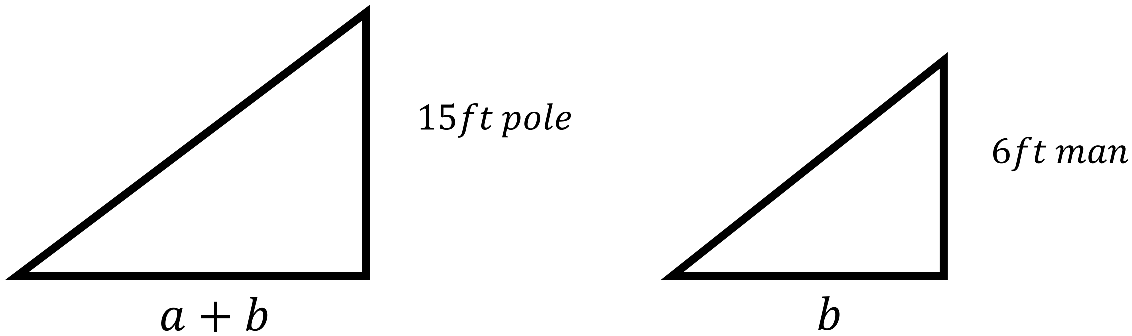
$$\theta' = \frac{-3}{50} \cdot \sin^2(\sin^{-1}\frac{1}{3})$$

$$\boldsymbol{\theta' = -0.0069\frac{rad}{s}}.$$

To solve the second problem, we will use **the properties of similar triangles**. First let's model the problem



Using this model, we can create two distinct but similar triangles:



The bigger of the two triangles represents the distance from pole to the man while the smaller of the two represents the man and the tip of his shadow. Our end goal is to determine **how fast the tip of the shadow** is moving if the man is 30ft from pole, so we will depict it with the following formula:

$$(a+b)'|_{a=30}$$

Using similar triangles, we will retain only one variable:

$$\frac{20}{6} = \frac{a+b}{b}$$

$$20b = 6(a+b)$$

$$14b = 6(a)$$

$$b = \frac{3}{7}a\,.$$

Since the bigger triangle includes the tip of the shadow, we will **substitute b for a on that triangle and take the derivative:**

$$a + b = a + \frac{3}{7}a$$

$$(a+b)' = \left(a + \frac{3}{7}a\right)'$$

$$\left(a + \frac{3}{7}a\right)' = \frac{10}{7}a'$$

since a' is 8 ft/s (the man's constant walking speed), we conclude that the tip of the shadow changes at a rate of:

$$\frac{10}{7}(8) = \frac{80}{7} \rightarrow \mathbf{11.43\ ft/s}$$

Mean Value Theorem

Let's practice using the Mean Value Theorem (MVT) with the following examples:

1. Find all possible c in [0, 4] satisfying the conclusion of the MVT for $f(x) = x\sqrt{2x+8}$
2. If f(1)= 10 and f'(x) ≥ 2 for 1 ≤ x ≤ 4 how small can f(4) possibly be

MVT explains that **if we are given a D*ifferentiable* and *Continuous* function, then there exists *at least* one point on that function whose slope is equal to the tangent line.** To solve problems using MVT we will use this general formula

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In the first example, we first **take the derivative of the given function and evaluate it at c:**

$$f'(c) = \sqrt{2c+8} + \frac{c}{\sqrt{2c+8}}$$

$$\frac{(\sqrt{2c+8})^2 + c}{\sqrt{2c+8}}$$

$$= \frac{3c+8}{\sqrt{2c+8}}$$

then we **set it equal to** $\boldsymbol{\frac{f(b)-f(a)}{b-a}}$

$$\frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(0)}{4}$$

$$\frac{16-0}{4} \rightarrow 4$$

$$\frac{3c+8}{\sqrt{2c+8}} = 4$$

and solve for c

$$\frac{3c+8}{\sqrt{2c+8}} - 4 = 0$$

$$\frac{3c + 8 - 4\sqrt{2c + 8}}{\sqrt{2c + 8}} = 0$$

$$3c + 8 - 4\sqrt{2c + 8} = 0 \ (since\ the\ demoninator\ can\ never\ be\ 0)$$

$$3c + 8 = 4\sqrt{2c + 8}$$

$$(3c + 8)^2 = \left[4\sqrt{2c + 8}\right]^2$$

$$9c^2 + 48c + 64 = 16(2c + 8)$$

$$9c^2 + 16c - 64 = 0$$

Apply the quadratic formula

$$\frac{-16 \pm \sqrt{16^2 - 4(9)(-64)}}{2(9)}$$

$$\frac{-16 \pm \sqrt{2560}}{18}$$

$$\frac{-16 \pm 16\sqrt{10}}{18} = \frac{\mathbf{-8 \pm 8\sqrt{10}}}{\mathbf{9}}$$

In the second example we are given a, b, f(a), and are asked to determine the smallest possible value of f(b). Appling the MVT formula yields:

$$\frac{f(4) - f(1)}{4 - 1} \geq 2$$

$$f(4) - 10 \geq 6$$

$$\boldsymbol{f(4) \geq 16}\,.$$

Business Applications

Let’s now examine some common business examples:

Given the following **total** cost function (in dollars) C(x)=$15{,}000 + 180x - 0.5x^2$, what is the expected cost to produce the 10th item? What is the actual cost of producing the 10th item?

To find the **expected cost** to produce the 10th item, we **find the derivative of the function and plug in 10:**

$$C'(\mathrm{x}) = 180 - \mathrm{x}$$

$$C'(10) = 180 - 10 \rightarrow \$\mathbf{170}.$$

To find the actual cost of producing the 10th item **we find the total cost of producing 10 total and the total cost to produce 9 items and subtract the two.**

$$c(10) - c(9) = 15{,}000 + 180(10) - 0.5(10)^2 - [15{,}000 + 180(9) - 0.5(9)^2]$$

$$16750 - 16579.5 = \$\mathbf{170.50}$$

Now let's take a look at this example:

The function: y(t)= $75t^2 + 3t + 10$ represents the total number of electric vehicles **t** years from now (in millions). Estimate the amount by which the number of vehicles will increase **from now** to the next 6 years

To solve this problem, we must take the derivative and interpret the variables. The derivative can be written as this:

$$\frac{dy}{dt} = (150t + 3)$$

dy represents the **change in number of vehicles** (what we want to solve for), and dt is the **change in time.** If we solve for dy, we'll get this

$$dy = (150t + 3)dt.$$

Since t represents the time from now, we know that 0 years have passed and that the change in years is 6. Therefore, we plug in 0 for t and 6 for dt.

$$dy = (100(0) + 3) \cdot 6.$$

$$\mathbf{18.}$$

So, we'd expect an increase of 18 million vehicles over the next 6 years.

Outro

Thank you for watching TutorTube! I hope you enjoyed this video. Please subscribe to our channel @UNTLC for more exciting videos! Check out the links in the description below for more information about The Learning Center and follow us on social media. See you next time!

References

Stewart, James *Calculus: Early Transcendentals*, 7th Ed., Brooks/Cole Cengage Learning, 2010