TutorTube: Intro to Integration: Definition and Approximations

Introduction

Hello! Welcome to TutorTube, where The Learning Center's Lead Tutors help you understand challenging course concepts with easy-to-understand videos. My name is Ebby, Lead Tutor for Math and Political Science. In today's video, we will have an introduction to Integration with the definition and approximation techniques. Let's get started!

Left Endpoint Rule

In our first example we will utilize the **Reimann Sums Approximation** and take left endpoints to estimate the area under the curve. The function we want to estimate is f(x)= 1/x from x=1 to x=2 using 4 rectangles. An appropriate first step is to sketch the graph. Then we can find delta x or **the width of each rectangle**. Delta x is always given by the following function:

$$\Delta x = \frac{b-a}{n} \text{ where } b \text{ is the upper bound}, a \text{ the lower, and } n \text{ number of rectangles}$$

Using the given information, we obtain the following for delta x:

$$\frac{2-1}{4} = \frac{1}{4}.$$

Now that we have the width of each rectangle, we need to obtain the heights. The heights **always depend on which endpoint you're taking (Left, Right, or even midpoints).** In this case we are taking left endpoints, so we take the height of the first rectangle which is x=1. Then we increment by delta x ($\frac{1}{4}$ in this case) until we reach 4 heights (1 for each rectangle):

$$f(1), f\left(1\frac{1}{4}\right), f\left(1\frac{1}{2}\right), f\left(1\frac{3}{4}\right) \text{ represent the heights of each rectangle}.$$

Now that we have both the width and heights of each rectangle, **we take the area of each rectangle and add them up** in order to complete the approximation. We denote this by

$$A = \Delta x \sum_{i=1}^{n} f(x_i).$$

Essentially, we are repeating the "base times height" formula for n number of rectangles. Using the heights that we obtained previously, we have the following function

$$A = \frac{1}{4}(\mathrm{f}(1) + \mathrm{f}(1.25) + \mathrm{f}(1.5) + \mathrm{f}(1.75)$$

Which simplifies to

$$\frac{1}{4}(1 + 0.8 + 0.6667 + 0.5714). \mathit{accurate\ to\ 4\ decimals\ or\ better}.$$

and finally, we have our approximation

$$\mathbf{0.7595}.$$

If we compare this to the value that we get from using technology (Calculator/Computer):

$$\int_{1}^{2} \frac{1}{x} dx \ \approx \ 0.6931$$

we see that it is an overestimate.

Right Endpoint Rule

Now let's estimate the area under sin(x) from 0 to $\boldsymbol{\frac{\pi}{2}}$ using 4 rectangles and right endpoints. Approximation via right endpoints follows almost the exact same procedure as left endpoints but with a slight modification. We again begin by determining delta x:

$$\Delta x = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$

Since we are using right endpoints this time, we will not choose the leftmost height first but rather the **right endpoint starting at $\boldsymbol{\frac{\pi}{8}}$.** Then we will increment by $\boldsymbol{\frac{\pi}{8}}$ to obtain the other 3 heights that we need:

$$\mathrm{f}\left(\frac{\pi}{8}\right), \mathrm{f}\left(\frac{\pi}{4}\right), \mathrm{f}\left(\frac{3\pi}{8}\right), \mathrm{f}\left(\frac{\pi}{2}\right) \ \mathit{are\ the\ heights}\ .$$

Now we estimate area under the curve using the same formula

$$A = \Delta x \sum f(x_i)$$

Which gives

$$\frac{\pi}{8}\left[\mathrm{f}\left(\frac{\pi}{8}\right) + \mathrm{f}\left(\frac{\pi}{4}\right) + \mathrm{f}\left(\frac{3\pi}{8}\right) + \mathrm{f}\left(\frac{\pi}{2}\right)\right]$$

$$\frac{\pi}{8}[0.3826 + 0.7071 + 0.9239 + 1]$$

And simplifies to

$$\mathbf{1.1834.}$$

Again, we compare this to integral solved using technology:

$$\int_0^{\frac{\pi}{2}} sinx = 1$$

And we see again that it is an overestimate.

Midpoint Rule

Now let's take a look at midpoint rule. We will use it to approximate

$$\int_0^1 \sqrt{x^3 + 1}\, dx$$

Using 5 rectangles. From now on we will utilize this standard form of writing integrals (the procedure doesn't change but writing the problem in this way is more indicative of what you'll see in the future). Let's now obtain delta x:

$$\Delta x = \frac{1 - 0}{5} = \frac{1}{5}$$

In order to obtain the heights using midpoint rule, we take the midpoint of each width. The 5 widths are between:

$$x = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$$

The first midpoint is between x = 0 and 1/5, so we use the **standard midpoint formula:** $\boldsymbol{\frac{a+b}{2}}$ to obtain

$$\frac{0 + {}^{1}/_{5}}{2} = \frac{1}{10}$$

The second midpoint follows the same procedure, this time between 1/5 and 2/5

$$\frac{{}^{1}/_{5} + {}^{2}/_{5}}{2} = \frac{3}{10}$$

Repeating this algorithm for the rest will yield:

$$f\left(\frac{1}{10}\right), f\left(\frac{3}{10}\right), f\left(\frac{1}{2}\right), f\left(\frac{7}{10}\right), f\left(\frac{9}{10}\right)$$

as the heights. Now, Using the area formula we obtain

$$\frac{1}{5}(1.0005 + 1.0134 + 1.0607 + 1.1589 + 1.3149) = \mathbf{1.1097}.$$

Again, with technology, we see that

$$\int_0^1 \sqrt{x^3 + 1}\, dx \cong 1.1114$$

Which means that the **Midpoint Rule underestimates in this case but is more accurate than taking Left or Right Endpoints.**

Definition of the integral

We use the definition of the integral to find the area under the curve. Like many definitions, we hardly ever use it to solve problems in practice. However, it is important to practice using it when a new concept is introduced in order to understand the premise(s) of the mathematical principles involved.

Let's take the following theorem

4 Theorem If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\, \Delta x$$

where $\Delta x = \dfrac{b - a}{n}$ and $x_i = a + i\, \Delta x$

Stewart, James *Calculus: Early Transcendentals*, 7th Ed., Brooks/Cole Cengage Learning, 2010 along with the following summations:

5 $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

6 $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

7 $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Stewart, James *Calculus: Early Transcendentals*, 7th Ed., Brooks/Cole Cengage Learning, 2010

And solve the following the integrals:

$$1.\ \int_0^1 (4+3x^2)dx$$

$$2.\ \int_1^4 (x^3 - 4x + 1)dx\,.$$

The theorem essentially describes taking an infinite number of **n rectangles** with **widths of** Δx and **heights of** $\boldsymbol{f x_i}$. Much like the approximations we discussed earlier, every rectangle has the same width but different heights. Each height is represented by $\boldsymbol{f x_i}$ where $\boldsymbol{x_i}$ **represents the** $\boldsymbol{i^{th}}$increment by $\Delta \boldsymbol{x}$**.** So, we are following essentially the exact same procedure as before only taking many more rectangles. Beginning with example 1, the first thing to do is to determine $\Delta \boldsymbol{x}$ which in this case would be

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}.$$

Since we are taking n number of rectangles. We can also write $\boldsymbol{x_i}$**:**

$$x_i = 0 + i\left(\frac{1}{n}\right) = \frac{1}{n}i\,.$$

Together, we can apply both to the integral formula

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x\,,$$

Doing so looks like this:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{1}{n}i\right)\cdot\frac{1}{n}$$

This is the form of the solution to the problem; our job now is to simplify and solve the summation problem using the 3 summations given above. For simplicity let's put $\frac{1}{n}$ on the outside since the summation doesn't depend on n:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{1}{n}i\right)\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{1}{n}i\right)$$

Next, we must **find $f\left(\frac{1}{n}i\right)$** by **plugging in** $\left(\frac{1}{n}i\right)$ into $(4+3x^2)$. This yields

$$f\left(\frac{1}{n}i\right) = 4 + 3\left(\frac{1}{n}i\right)^2 = 4 + 3\left(\frac{1}{n^2}i^2\right).$$

Now, we have:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} 4 + \frac{3}{n^2}i^2.$$

Now that we have addition in the summation, we can **split the sum over addition:**

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} 4 + \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{3}{n^2}i^2.$$

and simplify by taking $\frac{3}{n^2}$ out of the 2nd sum

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} 4 + \lim_{n\to\infty}\frac{3}{n^3}\sum_{i=1}^{n} i^2.$$

The first summation simplifies to 4n since there's no i incrementing it. For the 2nd summation, we'll use the fact that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

To simplify it. Now we have

$$\lim_{n\to\infty}\frac{1}{n}(4n) + \lim_{n\to\infty}\frac{3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

Which simplifies to

$$4 + \lim_{n\to\infty}\frac{3\cdot {}^{n}/_{n}\,{}^{(n+1)}/_{n}\,{}^{(2n+1)}/_{n}}{6}$$

$$4 + \lim_{n\to\infty} \frac{3 \cdot 1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$$

and finally, we have

$$4 + \lim_{n\to\infty} \frac{3 \cdot 1(1+0)(2+0)}{6}$$

4+1 = **5**

Now let's apply the same procedure for the second example. First find Δx and x_i

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_i = 1 + i\left(\frac{3}{n}\right)$$

Then plug them both into the formula:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{3}{n} i\right) \cdot \frac{3}{n}$$

evaluate $f(x_i)$ and factor out $\frac{3}{n}$ and

$$f\left(1+\frac{3}{n}i\right) = \left(1+\frac{3}{n}i\right)^3 - 4(1+\frac{3}{n}i) + 1$$

$$= \left(1 + \frac{6}{n}i + \frac{9}{n^2}i^2\right)\left(1+\frac{3}{n}i\right) - 4 - \frac{12}{n}i + 1$$

$$\left(1 + \frac{3}{n}i + \frac{6}{n}i + \frac{18}{n^2}i^2 + \frac{9}{n^2}i^2 + \frac{27}{n^3}i^3\right) - \frac{12}{n}i - 3.$$

$$\left(1 + \frac{9}{n}i + \frac{27}{n^2}i^2 + \frac{27}{n^3}i^3\right) - \frac{12}{n}i - 3$$

$$\frac{-3}{n}i + \frac{27}{n^2}i^2 + \frac{27}{n^3}i^3 - 2$$

We can split it up into 4 summations:

$$\lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{27}{n^3} i^3 + \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{27}{n^2} i^2 + \lim_{n\to\infty} \frac{-3}{n} \sum_{i=1}^{n} \frac{3}{n} i + \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} -2$$

Factor out the n's

$$\lim_{n\to\infty} \frac{81}{n^4} \sum_{i=1}^{n} i^3 + \lim_{n\to\infty} \frac{81}{n^3} \sum_{i=1}^{n} i^2 + \lim_{n\to\infty} \frac{-9}{n^2} \sum_{i=1}^{n} i + \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} -2$$

Evaluate the sums

$$\lim_{n\to\infty}\frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{4}+\lim_{n\to\infty}\frac{81}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\lim_{n\to\infty}\frac{-9}{n^2}\cdot\frac{n(n+1)}{2}-\lim_{n\to\infty}\frac{6n}{n}$$

And finally simplify

$$\frac{81(1)(1)}{4}+\frac{81(1)(2)}{6}-\frac{9(1)(1)}{2}-6=\mathbf{\frac{174}{4}}$$

Outro

Thank you for watching TutorTube! I hope you enjoyed this video. Please subscribe to our channel @UNTLC for more exciting videos! Check out the links in the description below for more information about The Learning Center and follow us on social media. See you next time!