TutorTube: Integration by Parts

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Introduction

Hello and welcome to TutorTube, where The Learning Center's Lead Tutors help you understand challenging course concepts with easy-to-understand videos. My name is Ebby, Lead Tutor for Math and Political Science. In today's video, we will explore integration by parts. Let's get started!

Review Previous Methods of Integration

Before beginning integration by parts (IBP), let's try some quick examples:

1. $\int \frac{3\ln^2 x}{x} dx$
2. $\int \cos(2x)\, dx$
3. $\int_5^7 x\sqrt[3]{x-5}$

In the first example, we will use **the u-sub method** because we have a function and its derivative represented in the integral. To make it clearer re-write the integral like this:

$$3\int \frac{1}{x}\ln^2 x\, dx$$

Now take u as ln(x) and du as $\frac{1}{x}dx$ and the integral will change to

$$3\int u^2 du\,.$$

Now apply the reverse power rule to evaluate the integral

$$3\left[\frac{1}{3}u^3\right]$$

then put it back in terms of x by using the original u-sub

$$\mathbf{\ln^3 x + C}\,.$$

For the second example we have to determine the trig value **whose derivative gives us cos,** and because of the chain rule **we have to divide by the derivative of the argument**. Since the derivative of sine is cosine the integral of cosine will be sine, which we will then divide by 2 because of chain rule:

$$\boldsymbol{\frac{1}{2}\sin(2x) + C}\,.$$

To solve the 3rd example, we will use **the u-sub method.** Let's take x-5 as our u:

$$u = x - 5$$
$$du = dx$$

The new integral in terms of u will now be:

$$\int_{u(5)}^{u(7)} x\sqrt[3]{u}\,du$$

we have an extra variable that's not in terms of u so **use the original u-sub and solve for x to make it in terms of u:**

$$u = x - 5 \rightarrow u + 5 = x$$
$$\int_{u(5)}^{u(7)} (u+5)\sqrt[3]{u}\,du$$

The new bounds are in terms of u so we will use the substitution to determine their values:

$$u(5) = 5 - 5 \ \rightarrow 0$$
$$u(7) = 7 - 5 \ \rightarrow 2$$

This yields

$$\int_{0}^{2} (u+5)u^{\frac{1}{3}}\,du.$$

Now we simplify by distributing

$$\int_0^2 u^{\frac{4}{3}}+5u^{\frac{1}{3}}\,du\ ,$$

apply reverse power rule

$$\left[\frac{u^{7/3}}{7/3}+\frac{5u^{4/3}}{4/3}\right]_0^2$$

and finally, apply FTC (Fundamental Theorem of Calculus)

$$\mathbf{\frac{3}{7}(2)^{\frac{7}{3}}+\frac{15}{4}(2)^{\frac{4}{3}}}\ \rightarrow \boldsymbol{about\ 11.61}$$

Integration by Parts Intro

Now that we have reviewed some important integration examples, let's take a look at some IBP examples. Integration by parts (IBP) is the process of finding the antiderivative of a function, by breaking the function into three major parts. Essentially, we are reversing the Product Rule. This splitting of the function is performed to enable us to find a solution to the problem. The **formula** that is required to perform integration by parts is:

$$\int udv = uv - \int vdu$$

Now, we consider a hierarchical method that tells us the best choice for u. This is known as ILATE (or LIATE), where:

$$L = logarithmic$$

$$I = Inverse\ trig$$

$$A = Algebraic$$

$$T = Trigonometric$$

$$E = Exponential$$

Using these principles, let's solve this integral:

$$\int (lnx)^2 \; dx \,.$$

This can be re-written as:

$$\int (\ln(x))(\ln(x)) \; dx.$$

Using the hierarchy, ln (x) is the logarithmic function, making it the best choice for u in this case. The rest of the function is represented by dv: ln(x)dx.

$$u = ln(x) \qquad\qquad dv = \ln(x)\, dx$$

Now we solve for du and v:

$$u = \ln(x)$$

$$du = \frac{1}{x}\, dx$$

$$dv = \ln(x) dx$$

$$v = \int \ln(x)\, dx$$

(You don't need to include "+C" here)

When we're utilizing the IBP method, we must **always look for the u and dv** where u is "the best choice" and dv typically being the rest of the integral. Now we will use the formula:

$$uv - \int vdu$$

To get the answer, which yields

$$(\ln(x))\left(\int \ln(x)\, dx\right) - \int \left(\int \ln(x)\, dx\right)\left(\frac{1}{x}\right) dx \,.$$

When applying it, we may not always know the $\int vdu$ and would have to apply the IBP process again. (Beware, it is NOT $\frac{1}{x}$!!). Thus, applying IBP to $v = \int \ln(x)\, dx$ follows the same procedure:

$u = \ln(x)$ $\qquad$ $dv = dx$

$du = \frac{1}{x}dx$ $\qquad$ $v = x$

Plugging u, v, dv, and du into the formula yields

$$(\ln(x))(x) - \int x\left(\frac{1}{x}dx\right)$$

Where we compute the integral by canceling out the x's:

$$(\ln(x))(x) - \int dx$$

and finally obtaining the integral:

$$xln(x) - x$$

Which we will substitute back into the original equation in order to finish off the problem:

$$\int (lnx)^2\ dx = \ln(x)\cdot(xln(x) - x) - \int (xln(x) - x)\left(\frac{1}{x}\right)dx\,.$$

Simplifying this yields

$$xln^2(x) - xln(x) - \int \ln(x) - 1dx$$

Solving the integral

$$xln^2(x) - xln(x) - [xln(x) - x - x] + C$$

And finally

$$\boldsymbol{xln^2(x) - 2xln(x) + 2x + C}$$

Integration by Parts

Now let's consider the following function:

$$\int_0^\infty ye^{-xy}dy\,.$$

The first thing to note is the variable of integration, even though we have two variables written the variable of integration is y (because there is a dy. If it was 'dx' the variable of integration would x). Since the variable of integration is y we would treat x like it's a constant. Next, we must re-write the improper integral by taking the limit at its undefined points. In this problem, the function is defined for all the values of the interval except at infinity, so we take the limit of that:

$$\lim_{a\to\infty}\int_0^a ye^{-xy}dy\,.$$

Now, we apply ILATE and take u to be y and dv to be e^{-xy} . Thus we have

$$u = y \qquad dv = e^{-xy}$$

$$du = dy \qquad v = \int e^{-xy}\,dy = -\frac{1}{x}e^{-xy}$$

Now we apply the formula and obtain

$$-\frac{y}{x}e^{-xy}\Big|_0^a - \int_0^a -\frac{1}{x}e^{-xy}\,dy$$

Since x is a variable, we can move it out of the integral

$$-\frac{y}{x}e^{-xy}\Big|_0^a + \frac{1}{x}\int_0^a e^{-xy}\,dy\,.$$

Then we evaluate both pieces

$$-\frac{a}{x}e^{-xa} + 0 + \frac{1}{x}\left[-\frac{1}{x}e^{-xy}\right]_0^a$$

Apply FTC

$$-\frac{a}{x}e^{-xa} + 0 - \frac{1}{x^2}\left(e^{-x(a)} - e^{-x(0)}\right)$$

Remember that we can re-write negative exponents:

$$-\frac{a}{xe^{xa}}-\frac{1}{x^2}\left(\frac{1}{e^{x(a)}}-\frac{1}{e^0}\right)$$

Now let the limit approach,

$$\lim_{a\to\infty}\left[-\frac{a}{xe^{xa}}-\frac{1}{x^2}\left(\frac{1}{e^{x(a)}}-\frac{1}{e^0}\right)\right]$$

remember that any non-zero value to power of 0 is 1

$$-\frac{\infty}{xe^{\infty}}-\frac{1}{x^2}\left(\frac{1}{e^{\infty}}-\frac{1}{1}\right)$$

The general rule of thumb is that exponential decay outlasts polynomial growth so the "$\frac{\infty}{xe^{\infty}}$" will tend to 0:

$$0-\frac{1}{x^2}(0-1)$$

$$\boldsymbol{\frac{1}{x^2}}.$$

Now let's consider the following function

$$\int_{\pi}^{\infty} e^{-x}\cos(3x)\,dx$$

Since this is another improper integral, we must apply the limit rule:

$$\lim_{a\to\infty}\int_{\pi}^{a} e^{-x}\cos(3x)\,dx$$

Now, let's take u as e^{-x} and dv as $\cos(3x)$:

$u = e^{-x}$ $\qquad\qquad$ $dv = \cos{(3x)}$

$du = -e^{-x}$ $\qquad\qquad$ $v = \frac{1}{3}\sin{(3x)}$

Then applying the $uv - \int vdu$ formula yields

$$\int_{\pi}^{a} e^{-x}\cos(3x)\,dx = \left[\frac{e^{-x}}{3}\sin(3x)\right]_{\pi}^{a} + \frac{1}{3}\int_{\pi}^{a} e^{-x}\sin(3x)\,dx\,.$$

Running through the IBP process again, this time with u as e^{-x} and dv as $\sin(3x)$ would yield

$u = e^{-x}$ $\qquad$ $dv = sin\,(3x)$

$du = -e^{-x}$ $\qquad$ $v = -\frac{1}{3}\cos(3x)$

Substituting this into what we have already yields

$$\int_{\pi}^{a} e^{-x}\cos(3x)\,dx = \left[\frac{e^{-x}}{3}\sin(3x)\right]_{\pi}^{a} + \frac{1}{3}\left[\left[-\frac{1}{3}\mathrm{e}^{-x}\cos\,(3x)\right]_{\pi}^{a} - \frac{1}{3}\int_{\pi}^{a} e^{-x}\cos(3x)\right]$$

The trick here is to recognize that we have $\int_{\pi}^{a} e^{-x}\cos(3x)\,dx$ on both sides. So, we can isolate it on one side. We'll do this by adding the $-\frac{1}{9}\int_{\pi}^{a} e^{-x}\cos(3x)dx$, to both sides which yields

$$\int_{\pi}^{a} e^{-x}\cos(3x)\,dx + \frac{1}{9}\int_{\pi}^{a} e^{-x}\cos(3x)dx = \left[\frac{e^{-x}}{3}\sin(3x)\right]_{\pi}^{a} - \left[\frac{\mathrm{e}^{-x}}{9}\cos\,(3x)\right]_{\pi}^{a}$$

Which simplifies to

$$\frac{10}{9}\int_{\pi}^{a} e^{-x}\cos(3x)dx = \left[\frac{e^{-x}}{3}\sin(3x)\right]_{\pi}^{a} - \left[\frac{\mathrm{e}^{-x}}{9}\cos\,(3x)\right]_{\pi}^{a}$$

Now apply FTC on the right-hand side

$$\frac{10}{9}\int_{\pi}^{a} e^{-x}\cos(3x)dx = \frac{1}{3}\left[e^{-a}\sin(3a) - e^{-\pi}\sin(3\pi)\right] - \frac{1}{9}\left[\mathrm{e}^{-a}\cos(3a) - \mathrm{e}^{-\pi}\cos\,(3\pi)\right].$$

Now, let the limit approach (remember that exponential decay will outlast the bounded cos value)

$$\frac{10}{9}\int_{\pi}^{a} e^{-x}\cos(3x)dx = \lim_{a\to\infty}\left[\frac{1}{3}\left(e^{-a}\sin(3a) - e^{\pi}\sin(3\pi)\right) - \frac{1}{9}\left(\mathrm{e}^{-a}\cos(3a) + \mathrm{e}^{-\pi}\cos\,(3\pi)\right]\right.$$

$$= \frac{1}{3}(0-0) - \frac{1}{9}(0 - \mathrm{e}^{-\pi}(-1))$$

$$\frac{10}{9}\int_{\pi}^{a} e^{-x}\cos(3x)dx = -\frac{1}{9}(\mathrm{e}^{-\pi})$$

$$\int_{\pi}^{a} e^{-x}\cos(3x)dx = -\frac{1}{10}(\mathrm{e}^{-\pi})$$

$$\boldsymbol{-\frac{1}{10}(\mathrm{e}^{-\pi})}$$

Outro

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