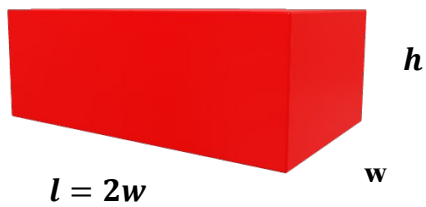
TutorTube Transcript: Optimization

Hello! Welcome to another edition of TutorTube. Today we will be talking about Optimization.

Example I: A rectangular storage container with an open top is to have a volume of **10** $\boldsymbol{m^3}$. The length of its base is twice the width. Material for the base costs **$12** per square meter. Material for the sides costs **$8** per square meter. Find the cost of materials for the cheapest such container. The container can be depicted as:



$\boldsymbol{l = 2w}$ comes from the problem and will help us create the equation that we need to optimize. In this problem we need three equations. First, the volume:

$$\boldsymbol{v = lwh}$$

Next, the area of the sides:

$$\boldsymbol{2lh + 2wh}$$

Third, the area of the base:

$$\boldsymbol{B = lw.}$$

The areas of the side and base are needed in order to determine the cost of materials. We will utilize the $\boldsymbol{l = 2w}$ expression to make every variable in terms of $\boldsymbol{w}$**.** Substituting the volume equation gives us:

$$\boldsymbol{v = 2w \cdot w \cdot h,}$$

Which simplifies to:

$$\boldsymbol{2w^2h.}$$

Since we know the volume is $\boldsymbol{10m^3}$**,** we can substitute **10** in for **v**:

$$\boldsymbol{10 = 2w^2h.}$$

Now, get **h** in terms of **w** by solving for **h**:

$$\boldsymbol{\frac{10}{2w^2} = h}$$

Which simplifies to

$$\boldsymbol{h = \frac{5}{w^2}}.$$

Using this information, we will put the side and base equations in terms of **h.** First, the area of the sides:

$$\boldsymbol{S = 2\,(2w)\left(\frac{5}{w^2}\right) + 2(w)\left(\frac{5}{w^2}\right).}$$

Simplifying yields:

$$\boldsymbol{\frac{20}{w} + \frac{10}{w} = \frac{30}{w}.}$$

Now the area of the base:

$$\boldsymbol{B = 2w \cdot w \rightarrow 2w^2\,.}$$

Now that we have both area equations in the same terms, we can determine the cost function. The cost function will be denoted by the letter $\boldsymbol{C}$**,** and represent the cost of materials for the base plus the cost of materials for the sides. Since the cost for the base is said to be **$12** per square meter and the cost for the sides **$8** per square meter, the cost function is:

$$\boldsymbol{C = \$12(2w^2) + \$8\left(\frac{30}{w}\right)}$$

Which can be simplified to be:

$$\boldsymbol{24w^2 + \frac{240}{w}\,.}$$

We would like to optimize the cost of materials which, by definition, means we will take derivative of the function and set it equal to zero. The derivative is:

$$\boldsymbol{C' = 48w - \frac{240}{w^2}.}$$

We must now set it equal to zero and factor:

$$\boldsymbol{48w\left(1 - \frac{5}{w^3}\right) = 0.}$$

Solving for $\boldsymbol{w}$ gives us the critical value:

$$\boldsymbol{\left(1 - \frac{5}{w^3}\right) = 0}$$

$$\boldsymbol{w = \sqrt[3]{5}\,.}$$

Finally, we will use the critical value to determine the optimal cost of materials by plugging into our *original cost function:*

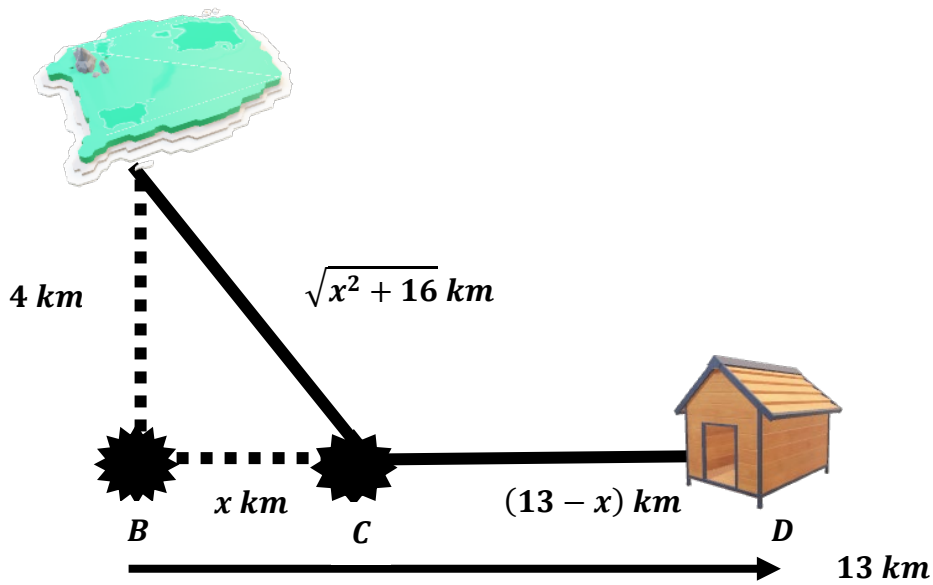
$$\boldsymbol{C = 24(\sqrt[3]{5})^2 + \frac{240}{\sqrt[3]{5}}.}$$

Thus, the cost of materials for the cheapest container is:

$\$210.53$.

Example II: Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than over land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is **4 km** from the nearest point **B** on a straight shoreline, flies to a point **C** on the shoreline, and then flies along the shoreline to its nesting area **D**. This Bird needs **1.5** times as much energy to fly over water as it does to fly over land. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. **Find how far point C is from B if Points B and D are 13 km apart**.

Given this information we can draw a picture of the bird's flight path:



The problem states that the island is **4km** from the shoreline and we created the variable **x** to represent the length from **b** to **c.** Therefore, we can get the hypotenuse

$$\sqrt{x^2+16}\ km$$

which represents the optimal distance the bird should travel over water, and also:

$$(13-x)\ km,$$

Which is the optimal distance the bird would travel over land. The problem also states that the bird needs **1.5** times as much energy to fly over water as it does over land. Therefore, we will let the energy constant the bird uses to fly over land be **k.** The energy that the bird uses to travel is a function of the distance that it covers, therefore we need to express both the energy that the bird uses to fly over water as well the energy used over land (shoreline). The energy it takes for the bird to fly over water can be depicted as:

$$\mathbf{1.5}\boldsymbol{k} \cdot \sqrt{\boldsymbol{x}^2 + \mathbf{16}}\ \boldsymbol{km},$$

And the energy expended over land as:

$$\boldsymbol{k}(\mathbf{13} - \boldsymbol{x})\ \boldsymbol{km}.$$

Both of these equations represent the optimal path that the bird should take, and the total energy expended, **E(x),** can be expressed as the sum of the two:

$$\boldsymbol{E}(\boldsymbol{x}) = \mathbf{1.5}\boldsymbol{k}\sqrt{\boldsymbol{x}^2 + \mathbf{16}}\ + \boldsymbol{k}(\mathbf{13} - \boldsymbol{x})$$

$$\boldsymbol{E}(\boldsymbol{x}) = \mathbf{1.5}\boldsymbol{k}(\boldsymbol{x}^2 + \mathbf{16})^{\frac{1}{2}} + \mathbf{13}\boldsymbol{k} - \mathbf{13}\boldsymbol{x}.$$

Now, we must optimize this path, which means taking the derivative and finding the critical value. 1st we take the derivative using chain rule and power rule:

$$\boldsymbol{E}(\boldsymbol{x}) = \mathbf{1.5}\boldsymbol{k}\left[\frac{\mathbf{1}}{\mathbf{2}} \cdot \mathbf{2x}(\boldsymbol{x}^2 + \mathbf{16})^{-\frac{1}{2}}\right] - \boldsymbol{k}.$$

Then simplify:

$$\boldsymbol{E}(\boldsymbol{x}) = \mathbf{1.5}\boldsymbol{k}\left[\mathbf{x}(\boldsymbol{x}^2 + \mathbf{16})^{-\frac{1}{2}}\right] - \boldsymbol{k}$$

$$= \mathbf{1.5}\boldsymbol{k}\left[\frac{\boldsymbol{x}}{(\boldsymbol{x}^2 + \mathbf{16})^{\frac{1}{2}}}\right] - \boldsymbol{k},$$

and set equal to 0:

$$\mathbf{1.5}\boldsymbol{k}\left[\frac{\boldsymbol{x}}{(\boldsymbol{x}^2 + \mathbf{16})^{\frac{1}{2}}}\right] - \boldsymbol{k} = \mathbf{0}.$$

Now, we must find the critical value by solving for x:

$$\mathbf{1.5}\boldsymbol{k}\left[\frac{\boldsymbol{x}}{(\boldsymbol{x}^2 + \mathbf{16})^{\frac{1}{2}}}\right] = \boldsymbol{k}$$

Divide both sides k:

$$\mathbf{1.5}\left[\frac{\boldsymbol{x}}{(\boldsymbol{x}^2 + \mathbf{16})^{\frac{1}{2}}}\right] = \mathbf{1}$$

$$\mathbf{1.5}[\boldsymbol{x}] = (\boldsymbol{x}^2 + \mathbf{16})^{\frac{1}{2}}.$$

Square both sides to get rid of the exponent:

$$\mathbf{1.5[x]^2 = (x^2 + 16)}$$

$$\mathbf{2.25x^2 - x^2 = 16}$$

$$\mathbf{1.25x^2 = 16}$$

$$\boldsymbol{x = \pm\sqrt{\frac{16}{1.25}}}.$$

Since the distance from **b** and **c** is between **0** and **13 km,** we will take the positive root which yields:

$$\mathbf{3.58}\ \boldsymbol{km}.$$

Thank you for watching another edition of TutorTube. Check out what else the Learning Center has to offer in the links below.

Summary: Calculus based optimization involves maximizing or minimizing functions that have "real world" applications. Due to the complexity of real-life situations, the most difficult aspect is often times finding the correct function to optimize in the 1st place. The two examples that we've covered today illustrate this very point. In the first example, we needed to derive the optimization function from 3 other equations related to the volume and surface areas of the box respectively. From there, we collected like terms in order to create our optimization function and found the critical value. Finally, plugging in the critical value back into the original optimization function gave us our desired cost. In example II, we sketched the bird's optimal path and used the auxiliary variable k (for its typical energy level) to create the optimization function. From there, we optimized the function and found the critical value in order to solve the problem.

Check out these links for more information on LC services!

Learning Center Website: https://learningcenter.unt.edu

Tutoring: https://appointments.unt.edu

Follow us on social media!

Facebook: www.facebook.com/UNTLC

Twitter: www.twitter.com/untlearning

Instagram: www.instagram.com/untlearningcenter